

Universality properties of Gelfand-Tsetlin patterns

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ABSTRACT. A standard Gelfand-Tsetlin pattern of depth n is a configuration of particles in $\{1, \dots, n\} \times \mathbb{R}$. For each $r \in \{1, \dots, n\}$, $\{r\} \times \mathbb{R}$ is referred to as the r^{th} level of the pattern. A standard Gelfand-Tsetlin pattern has exactly r particles on each level r , and particles on adjacent levels satisfy an interlacing constraint.

Probability distributions on the set of Gelfand-Tsetlin patterns of depth n arise naturally as distributions of eigenvalue minor processes of random Hermitian matrices of size n . We consider such probability spaces when the distribution of the matrix is unitarily invariant, prove a determinantal structure for a broad subclass, and calculate the correlation kernel.

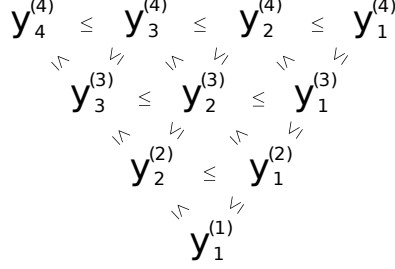
In particular we consider the case where the eigenvalues of the random matrix are fixed. This corresponds to choosing uniformly from the set of Gelfand-Tsetlin patterns whose n^{th} level is fixed at the eigenvalues of the matrix. Fixing $q_n \in \{1, \dots, n\}$, and letting $n \rightarrow \infty$ under the assumption that $\frac{q_n}{n} \rightarrow \alpha \in (0, 1)$ and the empirical distribution of the particles on the n^{th} level converges weakly, the asymptotic behaviour of particles on level q_n is relevant to free probability theory. Saddle point analysis is used to identify the set in which these particles behave asymptotically like a determinantal random point field with the Sine kernel.

1. Introduction

The spectrum of projections of random Hermitian matrices is an important object of study, both in free probability and in random matrix theory. For each $n \in \mathbb{N}$, let $\mathcal{H}_n \subset \mathbb{C}^{n \times n}$ be the set of $n \times n$ Hermitian matrices, and let $A_n \in \mathcal{H}_n$ be a random matrix whose distribution is unitarily invariant. For each $r \in \{1, \dots, n\}$, let $\pi_r \in \mathbb{C}^{n \times n}$ be the diagonal projection of rank r with the diagonal $(1, 1, \dots, 1, 0, 0, \dots, 0)$. Fix $q_n \in \{1, \dots, n\}$, and let $n \rightarrow \infty$ under the assumption that $\frac{q_n}{n} \rightarrow \alpha \in (0, 1)$ and the empirical eigenvalue distribution of A_n converges weakly to a compactly supported probability measure, μ . The asymptotic behaviour of the non-trivial eigenvalues of $\pi_{q_n} A_n \pi_{q_n}$ is of interest. In free probability, the asymptotic behaviour can be used to study the free additive convolution semi-group of μ (see Section 1.3 for a brief introduction, and Nica and Speicher, [22], for a more comprehensive reference). In this paper we identify the set in which the eigenvalues behave asymptotically like a determinantal random point field with the Sine kernel.

The non-trivial eigenvalues of projections can be considered as particles in a random interlaced system. For each $r \in \{1, \dots, n\}$, let $\mathcal{C}_r := \{(y_1^{(r)}, \dots, y_r^{(r)}) \in \mathbb{R}^r : y_1^{(r)} > \dots > y_r^{(r)}\}$, and $\lambda^{(r)} := (\lambda_1^{(r)}, \dots, \lambda_r^{(r)}) \in \overline{\mathcal{C}}_r$ be the non-trivial eigenvalues of $\pi_r A_n \pi_r$. Theorem 4.3.15 of Horn and Johnson, [14], then gives

$$(1.1) \quad \lambda_1^{(r+1)} \geq \lambda_1^{(r)} \geq \lambda_2^{(r+1)} \geq \lambda_2^{(r)} \geq \dots \geq \lambda_r^{(r)} \geq \lambda_{r+1}^{(r+1)},$$

FIGURE 1. A Gelfand-Tsetlin pattern, $(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)})$, of depth 4.

for all $r \in \{1, \dots, n-1\}$. We write $\lambda^{(r+1)} \succeq \lambda^{(r)}$ for all r , and say that the eigenvalues are *symmetrically interlaced*. Thus $(\lambda^{(1)}, \dots, \lambda^{(n)}) \in \overline{\text{GT}}_n$ where

$$(1.2) \quad \overline{\text{GT}}_n := \left\{ (y^{(1)}, \dots, y^{(n)}) \in \overline{\mathcal{C}}_1 \times \dots \times \overline{\mathcal{C}}_n : y^{(n)} \succeq y^{(n-1)} \succeq \dots \succeq y^{(1)} \right\}.$$

This is referred to as the set of *standard Gelfand-Tsetlin patterns of depth n* . Figure 1 gives an example of such a pattern.

The interlaced n -tuple $(\lambda^{(1)}, \dots, \lambda^{(n)}) \in \overline{\text{GT}}_n$ is referred to as the *eigenvalue minor process* of A_n . Letting μ_n be the distribution of $\lambda^{(n)} \in \overline{\mathcal{C}}_n$ (i.e. the eigenvalue distribution of A_n), and assuming that μ_n is supported on \mathcal{C}_n , it follows from Baryshnikov, [2], that $(\lambda^{(1)}, \dots, \lambda^{(n)})$ has distribution

$$(1.3) \quad d\nu_n[y^{(1)}, \dots, y^{(n)}] = \begin{cases} \prod_{i < j} \left(\frac{j-i}{y_i^{(n)} - y_j^{(n)}} \right) d\mu_n[y^{(n)}] dy^{(n-1)} \dots dy^{(1)} & ; y^{(n)} \in \mathcal{C}_n, \\ 0 & ; \text{otherwise,} \end{cases}$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\text{GT}}_n$, where $dy^{(r)}$ is Lebesgue measure on \mathbb{R}^r for each r . In the language of Baryshnikov, ν_n is the *uniform lift* of μ_n to $\overline{\text{GT}}_n$.

In Section 2 we consider the case where ν_n can be written in the form

$$(1.4) \quad d\nu_n[y^{(1)}, \dots, y^{(n)}] := \frac{1}{Z_n} \det \left[\phi_i(y_j^{(n)}) \right]_{i,j=1}^n dy^{(n)} dy^{(n-1)} \dots dy^{(1)},$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\text{GT}}_n$, where $\phi_1, \dots, \phi_n : \mathbb{R} \rightarrow \mathbb{R}$, and $Z_n > 0$ is a normalisation constant. Assuming integrability conditions on ϕ_1, \dots, ϕ_n , we prove that $(\overline{\text{GT}}_n, \nu_n)$ is a determinantal random point field and calculate the correlation kernel (see Section 1.1 for an introduction to determinantal random point fields). Perhaps the best studied example of such distributions is the eigenvalue minor process of the Gaussian unitary ensemble (GUE), which we discuss in more detail in Section 1.2. In this case, as we shall see, $\phi_i(y) = H_{n-i}(y)e^{-\frac{1}{2}y^2}$ for all $i \in \{1, \dots, n\}$ and $y \in \mathbb{R}$, where $H_i : \mathbb{R} \rightarrow \mathbb{R}$ is the Hermite polynomial of degree i .

Fixing $a, b \in \mathbb{R}$ with $a < b$, and $x^{(n)} \in \mathcal{C}_n \cap [a, b]^n$ for all $n \in \mathbb{N}$, consider the case where $\phi_i = \delta_{x_i^{(n)}}$ for all $i \in \{1, \dots, n\}$. Then, the measure in equation (1.4) is the distribution of the eigenvalue minor process of $U_n B_n U_n^*$, where $B_n \in \mathcal{H}_n$ is a fixed Hermitian matrix with eigenvalues $x^{(n)}$, and $U_n \in \mathbb{C}^{n \times n}$ is a random Unitary matrix chosen according to Haar measure. We are interested in the behaviour of $\lambda^{(q_n)}$ in the above asymptotic limit (i.e. $n \rightarrow \infty$ under the assumption that $\frac{q_n}{n} \rightarrow \alpha \in (0, 1)$ and the empirical distribution of $x^{(n)}$ converges weakly to μ).

In Section 1.3 we recall known results about the global asymptotic behaviour of $\lambda^{(q_n)}$. It follows from the interlacing constraint that the empirical distribution of $\lambda^{(q_n)}$ is supported on $[a, b]$. As we shall see, the expectation of the empirical distribution converges weakly to a measure μ_α on $[a, b]$ in the above asymptotic

limit. An expression for μ_α in terms of the free additive convolution semi-group of μ follows from the work of Voiculescu, [29], and a Lebesgue decomposition of μ_α can be characterised from the work of Belinschi, [3], [4].

In this paper we consider the local asymptotic behaviour of $\lambda^{(q_n)}$. The main result of this paper, described in detail in Section 1.4, can be summarised as follows:

Theorem 1.1. *For each $n \in \mathbb{N}$, let $K_n : \mathbb{R}^2 \rightarrow \mathbb{C}$ be the correlation kernel associated with $\lambda^{(q_n)}$. Then for all $c \in (a, b)$ contained in that subset of the support of μ_α on which μ_α is absolutely continuous with respect to Lebesgue measure,*

$$\lim_{n \rightarrow \infty} \frac{1}{n\rho_\alpha(c)} K_n \left(c + \frac{u}{n\rho_\alpha(c)}, c + \frac{v}{n\rho_\alpha(c)} \right) = \frac{\sin(\pi(v-u))}{\pi(v-u)},$$

where $\rho_\alpha(c)$ is the density of μ_α at the point c .

The limiting correlation kernel given above is referred to as the *Sine* kernel. This has been observed asymptotically in the spectrum of other ensembles of random matrices and in related systems (see, for example, [9], [15], [25]). Thus locally, as long as we avoid points where the non-trivial eigenvalues accumulate (i.e. atoms of μ_α), the eigenvalues are asymptotically distributed as a determinantal random point field with the Sine kernel. The strength of the above Theorem is that the asymptotic behaviour can be observed without needing specific information about μ . This is a generalisation of Collins, [8], who took B_n to be a projection of rank \tilde{q}_n with $\frac{\tilde{q}_n}{n} \rightarrow \beta \in (0, 1)$ as $n \rightarrow \infty$. In this case $\mu = (1 - \beta)\delta_0 + \beta\delta_1$. This result is recovered in Section 1.5.2.

Random systems with no obvious connection to random matrices sometimes give rise to related measures. Examples include the bead model (see Boutillier, [5]), random tilings (see, for example, [10], [17], [24]) and polynuclear growth (see Johansson, [16]). These models have subtle connections. For example Johansson and Nordenstam, [17], [23], consider random tilings of a hexagon with lozenges. Lozenges are shown to interlace, and, in the large hexagon limit, lozenges close to the boundary behave asymptotically like the eigenvalue minor process of the GUE.

The paper is structured as follows: Sections 1.1 and 1.2 motivate this topic by giving an introduction to determinantal random point fields, and by discussing the GUE case in greater detail. Section 1.3 recalls the known results regarding the global behaviour of $\lambda^{(q_n)}$ in the above asymptotic limit. The main result is stated in Section 1.4. Section 1.5 considers special cases of the measure μ .

Section 2 contains the initial results on the determinantal structure of the space $(\overline{\text{GT}}_n, \nu_n)$ when ν_n can be written in the form given in equation (1.4). We also calculate the correlation kernel. Though the main result of this section, Theorem 2.1, follows from the more general results of Defossez, [19], we give a simplified account. We obtain useful contour integral expressions for the correlation kernel in Proposition 2.4.

Section 3 contains a proof of the main result, Theorem 1.6. The asymptotic behaviour of the correlation kernel is obtained by performing a saddle point analysis on the contour integral expression for the kernel given in Proposition 2.4. Finally, in Section 4 we consider the case where the measure on the Gelfand-Tsetlin patterns is induced by the eigenvalue minor process of a Unitary invariant ensemble. In Section 4.1 we specialise to classical ensembles that satisfy a Rodrigues formula. We recover the correlation kernel of the eigenvalue minor process of the GUE obtained by Johansson and Nordenstam, [17] (see equation (1.9)).

1.1. Determinantal random point fields. The following is a brief introduction to determinantal random point fields. For a more complete treatment see Johansson, [18], and Soshnikov, [27].

Let E be a Polish space. Fix $N \in \mathbb{N} \cup \{\infty\}$, and let $\Omega \subset E^N$ be a space of configurations of N -particles of E . The case $N = \infty$ gives countable configurations. Denote each $\omega \in \Omega$ by $(\omega_1, \dots, \omega_N)$. We allow for multiple points, i.e., $\omega_i = \omega_j$ for $i \neq j$.

Given $\omega \in \Omega$, and a Borel set $B \subset E$, define $N_B(\omega) := \#\{i : \omega_i \in B\}$, the number of particles from ω contained in B . We call ω *locally finite* if $N_K(\omega)$ is finite for every compact set $K \subset E$. Assume Ω consists entirely of locally finite configurations. Given $m \leq N$, define $C_B^m \subset \Omega$ by $C_B^m := \{\omega \in \Omega : N_B(\omega) = m\}$. This is called a *cylinder set*. Let \mathcal{F} be the σ -algebra generated by the cylinder sets.

Definition 1.1. A random point field is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathbb{P} is a probability measure on (Ω, \mathcal{F}) .

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a random point field. For each $m \leq N$ define a measure, \mathbb{M}_m , on E^m by

$$(1.5) \quad \mathbb{M}_m[B] := \mathbb{E} \left[\sum_{1 \leq i_1 \neq \dots \neq i_m \leq N} 1_{\{\omega \in \Omega : (\omega_{i_1}, \dots, \omega_{i_m}) \in B\}} \right],$$

for any Borel subset $B \subset E^m$. We assume that \mathbb{M}_m is well-defined for all m , and $\mathbb{M}_m[B] < \infty$ whenever B is bounded. For each $m \leq N$, and each Borel subset $B \subset E^m$, $\mathbb{M}_m[B]$ is the expected number of m -tuples of particles from Ω that are contained in B . Also, for all $m \leq N$, and all disjoint bounded Borel sets $B_1, \dots, B_m \subset E$,

$$\mathbb{M}_m[B_1 \times \dots \times B_m] = \mathbb{E} \left[\prod_{k=1}^m N_{B_k} \right].$$

Letting μ be a reference measure on E , for example Lebesgue on \mathbb{R} , we make the following definition:

Definition 1.2. For any $m \leq N$, the Radon-Nikodym derivative of \mathbb{M}_m with respect to μ^m (if it exists) is referred to as the m^{th} correlation function of the random point field. That is, the m^{th} correlation function is the integrable function $\rho_m : E^m \rightarrow \mathbb{R}$ which satisfies

$$\mathbb{M}_m[B] = \int_B \rho_m(y_1, \dots, y_m) d\mu^m[y],$$

for all Borel subsets $B \subset E^m$.

This property is useful, for example, when calculating *last particle distributions*. That is, the distribution of the rightmost particle of random point fields over \mathbb{R} . See Johansson, [18], for more details.

Definition 1.3. A random point field is called *determinantal* if all correlation functions exist and there exists a function $K : E^2 \rightarrow \mathbb{C}$ for which

$$\rho_m(y_1, \dots, y_m) = \det[K(y_i, y_j)]_{i,j=1}^m,$$

for all $y_1, \dots, y_m \in E$ and $m \leq N$. K is called the *correlation kernel* of the field.

Remark 1.1. When \mathcal{F} and μ are ‘obvious’ they are not usually mentioned. For example when $E \subset \mathbb{R}$, \mathcal{F} is the Borel sigma-algebra and μ is Lebesgue measure. When $E \subset \mathbb{Z} \times \mathbb{R}$, $\mathcal{F} = \{A \times B : A \subset \mathbb{Z} \text{ and } B \subset \mathbb{R} \text{ is Borel}\}$ and μ is the direct product of the counting measure and Lebesgue measure.

Remark 1.2. Correlation kernels are not necessarily unique. For example when $E \subset \mathbb{R}$, another correlation kernel $J : E^2 \rightarrow \mathbb{C}$ can be defined by $J(u, v) := \frac{w(u)}{w(v)} K(u, v)$ for all $u, v \in \mathbb{R}$, where w is any non-zero complex function.

1.2. The eigenvalue minor process of the GUE. The GUE is the probability measure on \mathcal{H}_n given by

$$d\xi_n^{\text{GUE}}[H] := \frac{1}{Z_n} e^{-\frac{1}{2} \text{Tr} H^2} dH,$$

where $Z_n > 0$ is a normalisation constant, and dH is the Lebesgue measure

$$dH := \left(\prod_{i=1}^n d(H_{ii}) \right) \left(\prod_{j < k} d(\operatorname{Re} H_{jk}) d(\operatorname{Im} H_{jk}) \right).$$

A typical matrix chosen according to the GUE has diagonal elements given by independent standard Gaussians, and the real and imaginary part of the non-diagonal elements given by independent Gaussians with variance $\frac{1}{2}$.

Let $(\lambda^{(1)}, \dots, \lambda^{(n)}) \in \overline{\operatorname{GT}}_n$ be the eigenvalue minor process of the GUE, as discussed in Section 1. The distribution of $\lambda^{(n)} \in \overline{\mathcal{C}}_n$ (i.e. the distribution of the eigenvalues of the GUE) is given by (see for example Mehta, [20])

$$(1.6) \quad d\mu_n^{\operatorname{GUE}}[y] = \frac{1}{Z'_n} \Delta_n(y)^2 \left(\prod_{i=1}^n e^{-\frac{1}{2}y_i^2} \right) dy,$$

for all $y \in \overline{\mathcal{C}}_n$, where $Z'_n > 0$ is a normalisation constant, dy is Lebesgue measure on \mathbb{R}^n , and $\Delta_n : \mathbb{R}^n \rightarrow \mathbb{R}$ is the *Vandermonde* determinant

$$(1.7) \quad \Delta_n(y) := \prod_{1 \leq i < j \leq n} (y_i - y_j) = \det \left[y_i^{n-j} \right]_{i,j=1}^n.$$

Equation (1.3) thus implies that $(\lambda^{(1)}, \dots, \lambda^{(n)})$ has distribution

$$d\nu_n^{\operatorname{GUE}}[y^{(1)}, \dots, y^{(n)}] = \frac{1}{Z''_n} \Delta_n(y^{(n)}) \left(\prod_{i=1}^n e^{-\frac{1}{2}(y_i^{(n)})^2} \right) dy^{(n)} dy^{(n-1)} \dots dy^{(1)},$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\operatorname{GT}}_n$, where $Z''_n > 0$ is a normalisation constant, and $dy^{(r)}$ is Lebesgue measure on \mathbb{R}^r for each r .

Definition 1.1 implies that $(\overline{\mathcal{C}}_n, \mu_n^{\operatorname{GUE}})$ is a random point field on \mathbb{R} . Let $\{H_i\}_{i \geq 0}$ be the sequence of monic Hermite polynomials, i.e., for each $i, j \geq 0$, H_i and H_j have degree i and j respectively and satisfy

$$\int_{-\infty}^{\infty} H_i(y) H_j(y) e^{-\frac{1}{2}y^2} dy = \sqrt{2\pi i! j!} \delta_{ij}.$$

Equations (1.6) and (1.7) then give

$$d\mu_n^{\operatorname{GUE}}[y] = \frac{1}{Z'_n} \left(\det \left[H_{n-j}(y_i) e^{-\frac{1}{4}y_i^2} \right]_{i,j=1}^n \right)^2 dy,$$

for all $y \in \overline{\mathcal{C}}_n$. Proposition 2.11 of Johansson, [18], then shows that this field is determinantal with correlation kernel $K_n^{\operatorname{GUE}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$(1.8) \quad K_n^{\operatorname{GUE}}(u, v) = \sum_{i=0}^{n-1} \frac{1}{\sqrt{2\pi} i!} H_i(u) H_i(v) e^{-\frac{1}{4}(u^2+v^2)},$$

for all $u, v \in \mathbb{R}$.

More recently Johansson and Nordenstam, [17], showed a determinantal structure for $(\overline{\operatorname{GT}}_n, \nu_n^{\operatorname{GUE}})$. For simplicity of notation identify $\overline{\operatorname{GT}}_n$ with a space of configurations of $\frac{1}{2}n(n+1)$ particles on $\{1, \dots, n\} \times \mathbb{R}$ using the natural map from $\overline{\operatorname{GT}}_n$ to $(\{1, \dots, n\} \times \mathbb{R})^{\frac{1}{2}n(n+1)}$ given by

$$(y^{(1)}, \dots, y^{(n)}) \mapsto \left((1, y_1^{(1)}), (2, y_1^{(2)}), (2, y_2^{(2)}), (3, y_1^{(3)}), (3, y_2^{(3)}), (3, y_3^{(3)}), \dots \right),$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\operatorname{GT}}_n$. In words, the first particle of each configuration is contained in $\{1\} \times \mathbb{R}$, the next 2 particles are contained in $\{2\} \times \mathbb{R}$, next 3 in $\{3\} \times \mathbb{R}$ etc. Definition 1.1 thus implies that

$(\overline{\text{GT}}_n, \nu_n^{\text{GUE}})$ is a random point field on $\{1, \dots, n\} \times \mathbb{R}$. Johansson and Nordenstam, [17], show that this field is determinantal with correlation kernel $J_n^{\text{GUE}} : (\{1, \dots, n\} \times \mathbb{R})^2 \rightarrow \mathbb{R}$ given by

$$(1.9) \quad J_n^{\text{GUE}}((r, u), (s, v)) = \sum_{i=-\infty}^{-1} \frac{1}{\sqrt{2\pi}(i+s)!} H_{i+r}(u) H_{i+s}(v) e^{-\frac{1}{4}(u^2+v^2)} \\ + 1_{s>r} e^{\frac{1}{4}(u^2-v^2)} \left(\sum_{i=-s}^{-r-1} \frac{H_{i+s}(v)}{\sqrt{2\pi}(i+s)!} \int_u^\infty dx \frac{(x-u)^{-i-r-1}}{(-i-r-1)!} e^{-\frac{1}{2}x^2} - \frac{(v-u)^{s-r-1}}{(s-r-1)!} 1_{v>u} \right),$$

for all $r, s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$. Similar correlation kernels have been obtained for the eigenvalue minor processes of Jacobi and Laguerre ensembles (see equation (4.15) of Forrester and Nagao, [12]). Section 4.1 provides an alternative method for calculating these kernels.

As a final note we would like to point out some interesting asymptotics that are of relevance to our problem. For more information, see Anderson, Guionnet and Zeitouni, [1]:

Theorem 1.2. *Let μ_{sc} be the semicircle distribution, i.e., the distribution on \mathbb{R} with density $\rho_{sc} : \mathbb{R} \rightarrow \mathbb{R}$ given by*

$$(1.10) \quad \rho_{sc}(c) := \frac{1}{2\pi} \sqrt{4 - c^2} 1_{|c| \leq 2},$$

for all $c \in \mathbb{R}$. Then as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j^{(n)}/\sqrt{n}} \rightarrow \mu_{sc} \quad \text{almost surely,}$$

in the sense of weak convergence of measures.

Theorem 1.3. *For any $c \in (-2, 2)$, and any sequence $\{c_n\}_{n \geq 1} \subset \mathbb{R}$ with $\frac{c_n}{\sqrt{n}} \rightarrow c$,*

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{sc}(c)\sqrt{n}} K_n^{\text{GUE}} \left(c_n + \frac{u}{\rho_{sc}(c)\sqrt{n}}, c_n + \frac{v}{\rho_{sc}(c)\sqrt{n}} \right) = \frac{\sin(\pi(v-u))}{\pi(v-u)},$$

for all $u, v \in \mathbb{R}$.

For each $r \in \{1, \dots, n\}$, $J_n^{\text{GUE}}((r, \cdot), (r, \cdot)) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the correlation kernel for the particles on level r of the interlaced pattern (i.e. the eigenvalues of the sub-matrix of size r). Equations (1.8) and (1.9) give $J_n^{\text{GUE}}((r, \cdot), (r, \cdot)) = K_r^{\text{GUE}}$, and so the particles on level r are distributed as the eigenvalues of a randomly chosen GUE matrix of size r . Therefore, properly rescaled, the particles in the bulk on each level of the interlaced pattern behave asymptotically like a determinantal random point field with the Sine kernel.

Related systems of interlaced particles often display similar asymptotic behaviour. For example Boutilier, [5], studies the *bead model*, a probability measure on systems of interlaced particles on $\mathbb{Z} \times \mathbb{R}$. The particles on each thread (i.e. on $\{r\} \times \mathbb{R}$ for each r) form a determinantal random point field with the Sine kernel. In Metcalfe, O’Connell and Warren, [21], a circular analogue of this model is constructed.

1.3. Global asymptotic behaviour of the eigenvalues of random projections. Fix $a, b \in \mathbb{R}$ with $a < b$. For each $n \in \mathbb{N}$, fix $q_n \in \{1, \dots, n\}$, $x^{(n)} \in \mathcal{C}_n \cap [a, b]^n$ and $B_n \in \mathcal{H}_n$ with eigenvalues $x^{(n)}$. Let $U_n \in \mathbb{C}^{n \times n}$ be a random Unitary matrix chosen according to Haar measure, and let $(\lambda^{(1)}, \dots, \lambda^{(n)}) \in \overline{\text{GT}}_n$ be the eigenvalue minor process of $U_n B_n U_n^*$, as discussed in Section 1. In this section we recall known results about the behaviour of the empirical distribution of $\lambda^{(q_n)}$ under the following asymptotic limit:

Hypothesis 1.1. Let μ be a probability measure on \mathbb{R} which is not a point mass and with support, $\text{Supp}(\mu) \subset [a, b]$. Assume that, as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n)}} \rightarrow \mu,$$

in the sense of weak convergence of measures. Also assume that there exists an $\alpha \in (0, 1)$ for which $\frac{q_n}{n} \rightarrow \alpha$ as $n \rightarrow \infty$.

It follows from the interlacing constraint (see equation (1.1)) that $\lambda^{(q_n)} \in [a, b]^{q_n}$. Let $\mathbb{M}_1^{(n)}$ be the measure on $[a, b]$ of size q_n for which, for any $B \subset [a, b]$ measurable, $\mathbb{M}_1^{(n)}[B]$ is the expected number of eigenvalues from $\{\lambda_1^{(q_n)}, \dots, \lambda_{q_n}^{(q_n)}\}$ that are contained in B (see equation (1.5)). The following is a consequence of Voiculescu, [29]. For more information see Xu, [30], and Collins, [6], [7], [8]:

Lemma 1.4. Assuming hypothesis 1.1, $\frac{1}{n} \mathbb{M}_1^{(n)}$ converges weakly to μ_α , the measure on $[a, b]$ of size α given by

$$(1 - \alpha)\delta_0 + \mu_\alpha = ((1 - \alpha)\delta_0 + \alpha\delta_1) \boxtimes \mu,$$

where \boxtimes represents free multiplicative convolution.

For more information on free multiplicative convolution see Nica and Speicher, [22], lecture 14. Exercise 14.21 of this book gives an alternative expression for μ_α :

$$(1.11) \quad \mu_\alpha = \alpha D_\alpha(\mu^{\boxplus \alpha^{-1}}),$$

where D_α is the dilation operator that satisfies $D_\alpha \delta_a := \delta_{\alpha a}$ for all $a \in \mathbb{R}$, \boxplus represents free additive convolution, and $\{\mu^{\boxplus t}\}_{t \geq 1}$ is the free additive convolution semi-group of μ (i.e. $\mu^{\boxplus 1} = \mu$, $\mu^{\boxplus(s+t)} = (\mu^{\boxplus s}) \boxplus (\mu^{\boxplus t})$ for all $s, t \geq 1$, and the mapping $t \mapsto \mu^{\boxplus t}$ is continuous with respect to the weak* topology on probability measures).

A Lebesgue decomposition of μ_α follows from Belinschi, [3] (Theorem 4.1), [4] (Theorem 1.36):

Lemma 1.5. $\mu_\alpha = \mu_\alpha^{at} + \mu_\alpha^{ac} + \mu_\alpha^{sc}$ where

- (1) μ_α^{at} is an atomic measure with support $\text{Supp}(\mu_\alpha^{at}) = \{c \in [a, b] : \mu[\{c\}] > 1 - \alpha\}$. Moreover $\mu_\alpha^{at}[\{c\}] = \mu[\{c\}] - (1 - \alpha)$ for all $c \in \text{Supp}(\mu_\alpha^{at})$.
- (2) μ_α^{ac} is a non-zero measure which is absolutely continuous with respect to Lebesgue measure, and its density is analytic outside a closed set of Lebesgue measure zero.
- (3) μ_α^{sc} is singular continuous with respect to Lebesgue measure. Moreover the support of μ_α^{sc} has zero Lebesgue measure, and is included in the support of μ_α^{ac} .

1.4. Statement of the main result. Fix $a, b \in \mathbb{R}$ with $a < b$. For each $n \in \mathbb{N}$, fix $q_n \in \{1, \dots, n\}$, $x^{(n)} \in \mathcal{C}_n \cap [a, b]^n$ and $B_n \in \mathcal{H}_n$ with eigenvalues $x^{(n)}$. Let $U_n \in \mathbb{C}^{n \times n}$ be a random Unitary matrix chosen according to Haar measure, and let $(\lambda^{(1)}, \dots, \lambda^{(n)}) \in \overline{\text{GT}_n}$ be the eigenvalue minor process of $U_n B_n U_n^*$. Assume hypothesis 1.1. In this section we consider the local asymptotic behaviour of $\lambda^{(q_n)}$ as $n \rightarrow \infty$.

Equation (1.3) implies that $(\lambda^{(1)}, \dots, \lambda^{(n)})$ has distribution

$$(1.12) \quad d\nu_n[y^{(1)}, \dots, y^{(n)}] := \frac{1}{Z_n} \delta_{x^{(n)}}(y^{(n)}) dy^{(n)} dy^{(n-1)} \dots dy^{(1)},$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\text{GT}_n}$, where $Z_n > 0$ is a normalisation constant and $dy^{(r)}$ is Lebesgue measure on \mathbb{R}^r for each r . As in the GUE case (see Section 1.2), we identify $\overline{\text{GT}_n}$ with a space of configurations of $\frac{1}{2}n(n+1)$ particles on $\{1, \dots, n\} \times [a, b]$. Note, we restrict our attention to $[a, b]$ since $x^{(n)} \in [a, b]^n$, and so $\lambda^{(r)} \in [a, b]^r$ (for each r) by the interlacing constraint (see equation (1.1)). Definition 1.1 implies that

$(\overline{\text{GT}}_n, \nu_n)$ is a random point field on $\{1, \dots, n\} \times [a, b]$. Theorem 2.1 and remark 2.1 show that this field is determinantal with correlation kernel $K_n : (\{1, \dots, n\} \times [a, b])^2 \rightarrow \mathbb{C}$ given by

$$(1.13) \quad K_n((r, u), (s, v)) = \sum_{j=1}^n 1_{v \leq u < x_j^{(n)}} \frac{(x_j^{(n)} - u)^{n-r-1}}{(n-r-1)!} \frac{\partial^{n-s}}{\partial v^{n-s}} \prod_{i \neq j} \left(\frac{v - x_i^{(n)}}{x_j^{(n)} - x_i^{(n)}} \right) \\ - \sum_{j=1}^n 1_{v > u > x_j^{(n)}} \frac{(x_j^{(n)} - u)^{n-r-1}}{(n-r-1)!} \frac{\partial^{n-s}}{\partial v^{n-s}} \prod_{i \neq j} \left(\frac{v - x_i^{(n)}}{x_j^{(n)} - x_i^{(n)}} \right),$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in [a, b]$.

For each $n \in \mathbb{N}$, $K_n((q_n, \cdot), (q_n, \cdot)) : [a, b]^2 \rightarrow \mathbb{C}$ is the correlation kernel for $\lambda^{(q_n)}$, or equivalently the particles on level q_n of the Gelfand-Tsetlin pattern chosen according to the measure ν_n . We wish to establish a natural subset of (a, b) under which this kernel behaves asymptotically like the Sine kernel as $n \rightarrow \infty$. We define

$$(1.14) \quad A_\alpha := \{c \in (a, b) : \exists w \in \mathbb{C} \setminus \mathbb{R} \text{ with } wG_\mu(w+c) = 1-\alpha\},$$

where $G_\mu : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ is the Cauchy transform of μ (also known as the Stieltjes transform) given by

$$(1.15) \quad G_\mu(w) := \int_{-\infty}^{\infty} \frac{1}{w-x} \mu[dx],$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Proposition 1.7 gives a natural interpretation of A_α .

The main result (shown in section 3) can now be stated as follows:

Theorem 1.6. *Assume hypothesis 1.1. Then, given $c \in A_\alpha$, there exists a $w_{\alpha,c} \in \mathbb{C}$ with $\text{Im}(w_{\alpha,c}) > 0$ and*

$$\{w \in \mathbb{C} \setminus \mathbb{R} : wG_\mu(w+c) = 1-\alpha\} = \{w_{\alpha,c}, \overline{w_{\alpha,c}}\}.$$

Moreover for all $c \in A_\alpha$, and compact sets $U, V \subset \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \sup_{u \in U, v \in V} \left| \frac{(C_{\alpha,c})^{v-u}}{n\rho_\alpha(c)} K_n \left(\left(q_n, c + \frac{u}{n\rho_\alpha(c)} \right), \left(q_n, c + \frac{v}{n\rho_\alpha(c)} \right) \right) - \frac{\sin(\pi(v-u))}{\pi(v-u)} \right| = 0,$$

where $\rho_\alpha(c) := -\frac{1-\alpha}{\pi} \text{Im}(w_{\alpha,c}^{-1})$ and $C_{\alpha,c} := \exp \left(\pi \frac{\text{Re}(w_{\alpha,c}^{-1})}{\text{Im}(w_{\alpha,c}^{-1})} \right)$.

Natural interpretations exist for A_α and $\rho_\alpha : A_\alpha \rightarrow (0, \infty)$. Let μ_α be the measure on $[a, b]$ of size α given in Lemma 1.4, and let μ_α^{at} be its atomic part (see Lemma 1.5). Then, letting Supp represent support and $^\circ$ represent interior:

Proposition 1.7. *Assume hypothesis 1.1. Then A_α is open, $A_\alpha \cap \text{Supp}(\mu_\alpha^{\text{at}}) = \emptyset$, $A_\alpha \subset \text{Supp}(\mu_\alpha)^\circ$, and $\text{Supp}(\mu_\alpha)^\circ \setminus A_\alpha$ has Lebesgue measure zero. Moreover there exists an open subset of A_α , of equal Lebesgue measure, in which μ_α is absolutely continuous with respect to Lebesgue measure, and $\rho_\alpha(c)$ is the density of μ_α at c for each c in this set.*

To show this we consider the Cauchy transform and the \mathcal{R} -transform of μ_α . Letting ν be a probability measure on \mathbb{R} with compact support, the \mathcal{R} -transform of ν is the function, $\mathcal{R}_\nu : \mathbb{C} \rightarrow \mathbb{C}$, given by

$$\mathcal{R}_\nu(w) := \sum_{n \geq 0} \kappa_{n+1} w^n$$

for all $w \in \mathbb{C}$, where $\{\kappa_n\}_{n \geq 1}$ are the free cumulants of ν (see Nica and Speicher, [22], lecture 12, for more information). The following properties will be of use:

Lemma 1.8. *For any two probability measure ν, ξ on \mathbb{R} with compact support, and any $s \geq 1$, we have $\mathcal{R}_{\nu \boxplus s} = s\mathcal{R}_\nu$ and $\mathcal{R}_{\nu \boxplus \xi} = \mathcal{R}_\nu + \mathcal{R}_\xi$. Moreover, $G_\nu : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C} \setminus \mathbb{R}$ is invertible with inverse*

$$G_\nu^{-1}(w) = \mathcal{R}_\nu(w) + \frac{1}{w},$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 1.8 gives

$$w = G_\mu \left(\alpha (G_{\mu \boxplus \alpha^{-1}})^{-1}(w) + \frac{1 - \alpha}{w} \right),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Then, replacing w by $G_{\mu_\alpha}(w)$, and noting that $G_{\mu_\alpha}(w) = G_{\mu \boxplus \alpha^{-1}} \left(\frac{w}{\alpha} \right)$ (see equation (1.11)),

$$G_{\mu_\alpha}(w) = G_\mu \left(w + \frac{1 - \alpha}{G_{\mu_\alpha}(w)} \right),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 1.5 implies that there exists an open subset of $\text{Supp}(\mu_\alpha)^\circ$, of equal Lebesgue measure, in which μ_α is absolutely continuous with respect to Lebesgue measure and the density of μ_α is continuous. We extend the Cauchy transform, $G_{\mu_\alpha} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$, to this set by defining

$$(1.16) \quad G_{\mu_\alpha}(c) := \lim_{\epsilon \rightarrow 0_+} G_{\mu_\alpha}(c - i\epsilon),$$

for all c in the set. This is well-defined with $\frac{1}{\pi} \text{Im}(G_{\mu_\alpha}(c))$ equal to the density of μ_α at c . Therefore

$$(1.17) \quad G_{\mu_\alpha}(c) = G_\mu \left(c + \frac{1 - \alpha}{G_{\mu_\alpha}(c)} \right),$$

for all such c .

PROOF OF PROPOSITION 1.7: For each $c \in (a, b)$, define $f_{\alpha, c} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ by

$$(1.18) \quad f_{\alpha, c}(w) = G_\mu(w + c) - \frac{1 - \alpha}{w},$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Then $f_{\alpha, c}$ is analytic and $c \in A_\alpha$ if and only if roots of $f_{\alpha, c}$ exist (see equation (1.14)). Moreover, given $c \in A_\alpha$, there exists a $w_{\alpha, c} \in \mathbb{C}$ with $\text{Im}(w_{\alpha, c}) > 0$ and $\{w \in \mathbb{C} \setminus \mathbb{R} : f_{\alpha, c}(w) = 0\} = \{w_{\alpha, c}, \overline{w_{\alpha, c}}\}$ (see Theorem 1.6).

Fix $c \in A_\alpha$. Since $f_{\alpha, c}$ is a non-constant analytic function in $\mathbb{C} \setminus \mathbb{R}$ with $f_{\alpha, c}(w_{\alpha, c}) = 0$, there exists an $\epsilon \in (0, \text{Im}(w_{\alpha, c}))$ with $f_{\alpha, c}(w) \neq 0$ for all $w \in \overline{B}(w_{\alpha, c}, \epsilon) \setminus \{w_{\alpha, c}\}$. Thus, letting $\partial B(w_{\alpha, c}, \epsilon)$ be the boundary of $B(w_{\alpha, c}, \epsilon)$, the Bolzano-Weierstrass Theorem gives

$$\inf_{w \in \partial B(w_{\alpha, c}, \epsilon)} |f_{\alpha, c}(w)| > 0.$$

Rouché's Theorem (see Rudin, [26]) and equation (1.18) thus imply that there exists a $\delta > 0$ for which $f_{\alpha, c}$ and $f_{\alpha, y}$ have the same number of roots in $B(w_{\alpha, c}, \epsilon)$ for all $y \in (c - \delta, c + \delta)$. Therefore $(c - \delta, c + \delta) \subset A_\alpha$, and so A_α is open. Also equations (1.15) and (1.18) give

$$1 - \alpha = \int \frac{w_{\alpha, c}}{w_{\alpha, c} + c - x} d\mu[x].$$

Comparing real and imaginary parts gives

$$1 - \alpha = \int \frac{|w_{\alpha, c}|^2}{|w_{\alpha, c} + c - x|^2} d\mu[x] = \mu[\{c\}] + \int_{[a, b] \setminus \{c\}} \frac{|w_{\alpha, c}|^2}{|w_{\alpha, c} + c - x|^2} d\mu[x].$$

Thus, since μ is not a point mass (see hypothesis 1.1), $\mu[\{c\}] < 1 - \alpha$. Lemma 1.5 thus gives $c \notin \text{Supp}(\mu_\alpha^{\text{at}})$, and so $A_\alpha \cap \text{Supp}(\mu_\alpha^{\text{at}}) = \emptyset$.

We now show that $A_\alpha \subset \text{Supp}(\mu_\alpha)^\circ$. Fix $c \in A_\alpha$. Then, since A_α is open and $A_\alpha \cap \text{Supp}(\mu_\alpha^{\text{at}}) = \emptyset$, $[c - \delta, c + \delta]$ is a continuity set of μ_α for all $\delta > 0$ sufficiently small. Therefore Lemma 1.4 implies that

$$\mu_\alpha[c - \delta, c + \delta] = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{M}_1^{(n)}[c - \delta, c + \delta],$$

for all $\delta > 0$ sufficiently small, where $\mathbb{M}_1^{(n)}$ is the measure on $[a, b]$ of size q_n for which, for any $B \subset [a, b]$ measurable, $\mathbb{M}_1^{(n)}[B]$ is the expected number of eigenvalues from $\{\lambda_1^{(q_n)}, \dots, \lambda_{q_n}^{(q_n)}\}$ that are contained in B (see equation (1.5)). Then, since $K_n((q_n, \cdot), (q_n, \cdot)) : [a, b]^2 \rightarrow \mathbb{C}$ is the correlation kernel for $\lambda^{(q_n)}$, definitions 1.2 and 1.3 give

$$\mu_\alpha[c - \delta, c + \delta] = \lim_{n \rightarrow \infty} \int_{c-\delta}^{c+\delta} \frac{1}{n} K_n((q_n, y), (q_n, y)) dy,$$

for all $\delta > 0$ sufficiently small. Also, a slight extension of Theorem 1.6 (shown in the same way) gives

$$\lim_{n \rightarrow \infty} \sup_{y \in [c-\delta, c+\delta]} \left| \frac{1}{n\rho_\alpha(y)} K_n((q_n, y), (q_n, y)) - 1 \right| = 0,$$

for all $\delta > 0$ sufficiently small, where $\rho_\alpha(y) = -\frac{1-\alpha}{\pi} \text{Im}(w_{\alpha,y}^{-1})$, and so

$$\mu_\alpha[c - \delta, c + \delta] = \int_{c-\delta}^{c+\delta} \rho_\alpha(y) dy.$$

Thus, since $\rho_\alpha(y) > 0$ for all y , $\mu_\alpha[c - \delta, c + \delta] > 0$ for all $\delta > 0$ sufficiently small, and so $c \in \text{Supp}(\mu_\alpha)$. This is true for all $c \in A_\alpha$, and A_α is open, and so $A_\alpha \subset \text{Supp}(\mu_\alpha)^\circ$.

We now show that $\text{Supp}(\mu_\alpha)^\circ \setminus A_\alpha$ has Lebesgue measure zero. Lemma 1.5 implies that there exists an open subset of $\text{Supp}(\mu_\alpha)^\circ$, of equal Lebesgue measure, in which μ_α is absolutely continuous with respect to Lebesgue measure and the density of μ_α is continuous. For all c in this set, $G_{\mu_\alpha}(c)$ is well-defined and $\frac{1}{\pi} \text{Im}(G_{\mu_\alpha}(c))$ is the density of μ_α at c (see equation (1.16)). For all such c , equations (1.17) and (1.18) show that $f_{\alpha,c}$ has a root in $\mathbb{C} \setminus \mathbb{R}$ given by

$$\frac{1 - \alpha}{G_{\mu_\alpha}(c)}.$$

Thus all such c are in A_α , and so $\text{Supp}(\mu_\alpha)^\circ \setminus A_\alpha$ has Lebesgue measure zero. It remains to show that $\rho_\alpha(c)$ equals $\frac{1}{\pi} \text{Im}(G_{\mu_\alpha}(c))$ for all such c , the density of μ_α at c . This follows by noting that $\overline{w_{\alpha,c}} = \frac{1-\alpha}{G_{\mu_\alpha}(c)}$ (recall that there exists a $w_{\alpha,c} \in \mathbb{C}$ with $\text{Im}(w_{\alpha,c}) > 0$ and $\{w \in \mathbb{C} \setminus \mathbb{R} : f_{\alpha,c}(w) = 0\} = \{w_{\alpha,c}, \overline{w_{\alpha,c}}\}$) and $\rho_\alpha(c) = -\frac{1-\alpha}{\pi} \text{Im}(w_{\alpha,c}^{-1})$. \square

1.5. Examples. In this section we examine Theorem 1.6 in some special cases:

1.5.1. Semicircle distribution. Fix $\alpha \in (0, 1)$ and let μ be the semicircle distribution given in equation (1.10). Using the well known formula for the Cauchy transform of this distribution (see, for example, Anderson, Guionnet and Zeitouni, [1]), it follows from equation (1.14) that

$$A_\alpha = \left\{ c \in (-2, 2) : \exists w \in \mathbb{C} \setminus \mathbb{R} \text{ with } 1 - \frac{2(1-\alpha)}{w(w+c)} = \sqrt{1 - 4(w+c)^{-2}} \right\},$$

where we define $\sqrt{re^{i\theta}} := \sqrt{r}e^{i\frac{\theta}{2}}$ for all $r \geq 0$ and $\theta \in (-\pi, \pi]$. Then $A_\alpha = (-2\sqrt{\alpha}, 2\sqrt{\alpha})$ and $\{w \in \mathbb{C} \setminus \mathbb{R} : 1 - \frac{2(1-\alpha)}{w(w+c)} = \sqrt{1 - 4(w+c)^{-2}}\} = \{w_{\alpha,c}, \overline{w_{\alpha,c}}\}$ for all $c \in (-2\sqrt{\alpha}, 2\sqrt{\alpha})$, where

$$w_{\alpha,c} = \frac{(1-\alpha)c + (1-\alpha)\sqrt{4\alpha - c^2}i}{2\alpha}.$$

The density in Theorem 1.6 is given by

$$\rho_\alpha(c) = \sqrt{\alpha} \rho_{\text{sc}} \left(\frac{c}{\sqrt{\alpha}} \right),$$

for all $c \in (-2\sqrt{\alpha}, 2\sqrt{\alpha})$, where ρ_{sc} is the density of the semi-circle distribution.

1.5.2. A measure with two atoms. Fix $\alpha \in (0, 1)$, $\beta \in (0, 1)$ and $\mu := (1 - \beta)\delta_0 + \beta\delta_1$. It follows from equations (1.14) and (1.15) that

$$A_\alpha = \{c \in (0, 1) : \exists w \in \mathbb{C} \setminus \mathbb{R} \text{ with } \alpha(w + c)^2 + (\beta - \alpha - c)(w + c) + (1 - \beta)c = 0\}.$$

The discriminant of the quadratic polynomial is $(c - c_{\alpha,\beta}^-)(c - c_{\alpha,\beta}^+)$, where

$$c_{\alpha,\beta}^\pm := \left(\sqrt{(1 - \alpha)\beta} \pm \sqrt{\alpha(1 - \beta)} \right)^2.$$

Note that $0 < c_{\alpha,\beta}^- < c_{\alpha,\beta}^+ < 1$, and so $A_\alpha = (c_{\alpha,\beta}^-, c_{\alpha,\beta}^+)$. Moreover $\{w \in \mathbb{C} \setminus \mathbb{R} : \alpha(w + c)^2 + (\beta - \alpha - c)(w + c) + (1 - \beta)c = 0\} = \{w_{\alpha,c}, \overline{w_{\alpha,c}}\}$ for all $c \in (c_{\alpha,\beta}^-, c_{\alpha,\beta}^+)$, where

$$w_{\alpha,c} := -c + \frac{-\beta + \alpha + c \pm i\sqrt{(c - c_{\alpha,\beta}^-)(c_{\alpha,\beta}^+ - c)}}{2\alpha}.$$

The density in Theorem 1.6 is given by

$$\rho_\alpha(c) = \frac{\sqrt{(c - c_{\alpha,\beta}^-)(c_{\alpha,\beta}^+ - c)}}{2\pi c(1 - c)},$$

for all $c \in (c_{\alpha,\beta}^-, c_{\alpha,\beta}^+)$. This recovers the result of Collins, [8], who took $B_n \in \mathcal{H}_n$ to be a projection of rank \tilde{q}_n with $\frac{\tilde{q}_n}{n} \rightarrow \beta \in (0, 1)$ as $n \rightarrow \infty$. Collins computed the asymptotics by showing that $\pi_{q_n} U_n B_n U_n^* \pi_{q_n}$ is distributed according to a Jacobi ensemble of parameters $(q_n, n - \tilde{q}_n - q_n, \tilde{q}_n - q_n)$, and employing known asymptotic properties of Jacobi polynomials. Another example in which similar asymptotics arise is the discrete planar bead model examined by Fleming, Forrester, and Nordenstam, [11].

1.5.3. A measure with three atoms. Fix $\alpha \in (0, 1)$ and $\mu := \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$. It follows from equations (1.14) and (1.15) that

$$A_\alpha = \left\{ c \in (-1, 1) : \exists w \in \mathbb{C} \setminus \mathbb{R} \text{ with } \alpha(w + c)^3 - c(w + c)^2 + \left(\frac{2}{3} - \alpha \right) (w + c) + \frac{c}{3} = 0 \right\}.$$

The discriminant of the cubic polynomial is $\frac{4}{3}(c^2 - c_\alpha^-)(c^2 - c_\alpha^+)$, where

$$c_\alpha^\pm := \frac{3}{8} \left(g_\alpha \pm \sqrt{(g_\alpha)^2 + \frac{64}{3} \left(\frac{2}{3} - \alpha \right)^3 \alpha} \right), \quad g_\alpha := 3\alpha^2 + 6 \left(\frac{2}{3} - \alpha \right) \alpha - \left(\frac{2}{3} - \alpha \right)^2.$$

Note that $0 < c_\alpha^- < c_\alpha^+ < 1$ when $\alpha \in (\frac{2}{3}, 1)$, $c_\alpha^- = 0$ and $c_\alpha^+ = 1$ when $\alpha = \frac{2}{3}$, and $c_\alpha^- < 0 < c_\alpha^+ < 1$ when $\alpha \in (0, \frac{2}{3})$. It thus follows that $A_\alpha = (-\sqrt{c_\alpha^+}, -\sqrt{c_\alpha^-}) \cup (\sqrt{c_\alpha^-}, \sqrt{c_\alpha^+})$ for all $\alpha \in [\frac{2}{3}, 1)$, and $A_\alpha = (-\sqrt{c_\alpha^+}, \sqrt{c_\alpha^+})$ for all $\alpha \in (0, \frac{2}{3})$. Moreover, $\{w \in \mathbb{C} \setminus \mathbb{R} : \alpha(w + c)^3 - c(w + c)^2 + (\frac{2}{3} - \alpha)(w + c) + \frac{c}{3} = 0\} = \{w_{\alpha,c}, \overline{w_{\alpha,c}}\}$ for all $c \in A_\alpha$, where $w_{\alpha,c}$ is the root of the cubic in the upper half complex plane.

2. Determinantal structure of Gelfand-Tsetlin patterns

Define a probability measure on $\overline{\text{GT}}_n$, the set of Gelfand-Tsetlin patterns of depth n , by

$$(2.1) \quad d\nu_n[y^{(1)}, \dots, y^{(n)}] := \frac{1}{Z_n} \det \left[\phi_i(y_j^{(n)}) \right]_{i,j=1}^n dy^{(n)} dy^{(n-1)} \dots dy^{(1)},$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\text{GT}}_n$, where $Z_n > 0$ is a normalisation constant, $dy^{(r)}$ is Lebesgue measure on \mathbb{R}^r for each r , and $\phi_1, \dots, \phi_n : \mathbb{R} \rightarrow \mathbb{R}$ are such that the integrals in Theorem 2.1 are well-defined and finite. In this section we prove a determinantal structure for the space $(\overline{\text{GT}}_n, \nu_n)$. Though the main result of this section, Theorem 2.1, can be deduced from the more general results of Defosseux, [19], we give a simplified proof with an alternative expression for the correlation kernel.

Remark 2.1. *The measure in equation (1.12) can be written in the above form by taking $\phi_i = \delta_{x_i^{(n)}}$ for all $i \in \{1, \dots, n\}$. As we shall see in Section (4), the measure induced by the eigenvalue minor process of UIEs can also be written in this form.*

For technical reasons we consider a subset of $\overline{\text{GT}}_n$ on which the measure in equation (2.1) is supported. We say that a pair $(y^{(r)}, y^{(r+1)}) \in \mathcal{C}_r \times \mathcal{C}_{r+1}$ is *asymmetrically interlaced* if

$$y_1^{(r+1)} > y_1^{(r)} \geq y_2^{(r+1)} > y_2^{(r)} > \dots > y_r^{(r)} \geq y_{r+1}^{(r+1)}.$$

We denote this by $y^{(r+1)} \succ y^{(r)}$. Also, for each $n \geq 1$, define $\text{GT}_n \subset \mathcal{C}_1 \times \dots \times \mathcal{C}_n$ by

$$\text{GT}_n := \left\{ (y^{(1)}, \dots, y^{(n)}) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_n : y^{(n)} \succ y^{(n-1)} \succ \dots \succ y^{(1)} \right\}.$$

Comparing with $\overline{\text{GT}}_n$ (see equation (1.2)), $\text{GT}_n \subset \overline{\text{GT}}_n$ is the set of Gelfand-Tsetlin patterns of depth n with distinct particles and for which particles on neighbouring levels satisfy the asymmetric interlacing constraint. It is easy to see that ν_n is supported on GT_n .

As in Section 1.2, we identify GT_n with a space of configurations of $\frac{1}{2}n(n+1)$ particles on $\{1, \dots, n\} \times \mathbb{R}$. Definition 1.1 thus implies that (GT_n, ν_n) is a random point field on $\{1, \dots, n\} \times \mathbb{R}$. We shall prove the following:

Theorem 2.1. *Define $\Phi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ by*

$$(2.2) \quad \Phi_n(y) := \left(\prod_{k=1}^n \phi_k(y_k) \right) \Delta_n(y),$$

for $y \in \mathbb{R}^n$. Also define $B_n := \int_{\mathbb{R}^n} dy \Phi_n(y)$. Finally, letting \mathfrak{S}_n be the set of permutations of $\{1, \dots, n\}$, define $\mathfrak{S}_n \mathcal{C}_n := \bigcup_{\sigma \in \mathfrak{S}_n} \sigma(\mathcal{C}_n)$. Then $B_n \neq 0$, and the random point field (GT_n, ν_n) is determinantal with correlation kernel $K_n : (\{1, \dots, n\} \times \mathbb{R})^2 \rightarrow \mathbb{R}$ which satisfies

$$\begin{aligned} K_n((r, u), (s, v)) &= \frac{1}{B_n} \int_{\mathfrak{S}_n \mathcal{C}_n} dy \Phi_n(y) \sum_{j=1}^n 1_{v \leq u < y_j} \frac{(y_j - u)^{n-r-1}}{(n-r-1)!} \frac{\partial^{n-s}}{\partial v^{n-s}} \prod_{i \neq j} \left(\frac{v - y_i}{y_j - y_i} \right) \\ &\quad - \frac{1}{B_n} \int_{\mathfrak{S}_n \mathcal{C}_n} dy \Phi_n(y) \sum_{j=1}^n 1_{v > u \geq y_j} \frac{(y_j - u)^{n-r-1}}{(n-r-1)!} \frac{\partial^{n-s}}{\partial v^{n-s}} \prod_{i \neq j} \left(\frac{v - y_i}{y_j - y_i} \right), \end{aligned}$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$.

In order to show this we consider a related measure on systems of interlaced particles with the same number of *indistinguishable* particles on each level. Given $z, z' \in \mathfrak{S}_n \mathcal{C}_n$ with $z \in \sigma^{-1}(\mathcal{C}_n)$ and $z' \in \tau^{-1}(\mathcal{C}_n)$ some $\sigma, \tau \in \mathfrak{S}_n$, we say that the pair (z, z') is *interlaced* if

$$z'_{\tau(1)} > z_{\sigma(1)} \geq z'_{\tau(2)} > z_{\sigma(2)} \geq \dots \geq z'_{\tau(n)} > z_{\sigma(n)}.$$

Let $F_n \subset (\mathfrak{S}_n \mathcal{C}_n)^2$ be the set of all interlaced pairs. A nice characterisation of this type of interlacing is given in Warren, [31]: Given $z, z' \in \mathfrak{S}_n \mathcal{C}_n$ with $z \in \sigma^{-1}(\mathcal{C}_n)$ and $z' \in \tau^{-1}(\mathcal{C}_n)$ some $\sigma, \tau \in \mathfrak{S}_n$,

$$(2.3) \quad 1_{(z, z') \in F_n} = \det \left[1_{z'_{\tau(k)} > z_{\sigma(j)}} \right]_{j, k=1}^n.$$

Fix $n \in \mathbb{N}$, $M > 0$ and $(c_1, \dots, c_n) \in \mathcal{C}_n$ with $c_1 = -M$. Consider the space $(\mathbb{R}^n)^n$, interpreted as the set of configurations of n^2 particles in \mathbb{R}^n with exactly n particles in each \mathbb{R} . Denoting elements of this space by $\bar{z} := (z^{(1)}, \dots, z^{(n)})$, let $E \subset (\mathbb{R}^n)^n$ be the set of configurations for which

- $M \geq z_j^{(n)} \geq -M$ for all $j \in \{1, \dots, n\}$,
- $z_{\tau(j)}^{(1)} = c_j$ for all $j \in \{2, \dots, n\}$ whenever $z^{(1)} \in \tau^{-1}(\mathcal{C}_n)$ some $\tau \in \mathfrak{S}_n$,
- $(z^{(r)}, z^{(r+1)}) \in F_n$ for all $r \in \{1, \dots, n-1\}$.

Choosing $M > 0$ sufficiently large, we can define the measure, ξ_n , on $(\mathbb{R}^n)^n$ by

$$(2.4) \quad d\xi_n[\bar{z}] := \frac{1}{(n!)^n Z} \begin{cases} \det [\phi_i(z_{\sigma(j)}^{(n)})]_{i, j=1}^n dz^{(n)} dz^{(n-1)} \dots dz^{(1)} & ; \bar{z} \in E \text{ with } z^{(n)} \in \sigma^{-1}(\mathcal{C}_n), \\ 0 & ; \bar{z} \in (\mathbb{R}^n)^n \setminus E. \end{cases}$$

where $Z > 0$ is a normalisation constant, and $dz^{(r)}$ is the Lebesgue measure on \mathbb{R}^n for each r .

We identify $(\mathbb{R}^n)^n$ with a space of configurations of n^2 particles on $\{1, \dots, n\} \times \mathbb{R}$ using the natural map from $(\mathbb{R}^n)^n$ to $(\{1, \dots, n\} \times \mathbb{R})^{n^2}$ given by

$$(z^{(1)}, \dots, z^{(n)}) \mapsto \left((1, z_1^{(1)}), \dots, (1, z_n^{(1)}), (2, z_1^{(2)}), \dots, (2, z_n^{(2)}), (3, z_1^{(3)}), \dots, (3, z_n^{(3)}), \dots \right),$$

for all $(z^{(1)}, \dots, z^{(n)}) \in (\mathbb{R}^n)^n$. In words, the first n particles of each configuration are contained in $\{1\} \times \mathbb{R}$, the next n particles are contained in $\{2\} \times \mathbb{R}$, the next n in $\{3\} \times \mathbb{R}$ etc. Definition 1.1 thus implies that $((\mathbb{R}^n)^n, \xi_n)$ is a random point field on $\{1, \dots, n\} \times \mathbb{R}$. We now show this field is determinantal and calculate the correlation kernel.

Lemma 2.2. Define $\phi_{0,1} : \{1, \dots, n\} \times \mathbb{R} \rightarrow \mathbb{R}$, $\phi_{1,2}, \phi_{2,3}, \dots, \phi_{n-1,n} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi_{n,n+1} : \mathbb{R} \times \{1, \dots, n\} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \phi_{0,1}(i, v) &:= \begin{cases} 1_{M \geq v \geq -M} & ; i = 1, \\ \delta_{c_i}(v) & ; i \in \{2, \dots, n\}, \end{cases} \\ \phi_{r,r+1}(u, v) &:= 1_{M \geq v > u \geq c_n} \text{ for all } r \in \{1, \dots, n-1\}, \\ \phi_{n,n+1}(u, j) &:= \phi_j(u) 1_{M \geq u \geq -M}. \end{aligned}$$

Also define $z_j^{(0)} = z_j^{(n+1)} := j$ for all $j \in \{1, \dots, n\}$. Then for all $\bar{z} \in (\mathbb{R}^n)^n$,

$$(2.5) \quad d\xi_n[\bar{z}] = \frac{1}{(n!)^n Z} \prod_{r=0}^n \det [\phi_{r,r+1}(z_j^{(r)}, z_k^{(r+1)})]_{j,k=1}^n dz^{(n)} dz^{(n-1)} \dots dz^{(1)}.$$

PROOF. Fixing $\bar{z} \in (\mathcal{C}_n)^n$, it follows from the definition E that,

$$1_E(\bar{z}) = 1_{M \geq z_1^{(1)} \geq -M} \left(\prod_{j=2}^n \delta_{c_j}(z_j^{(1)}) \right) \left(\prod_{r=1}^{n-1} 1_{(z^{(r)}, z^{(r+1)}) \in F_n} \right) \left(\prod_{i=1}^n 1_{M \geq z_i^{(n)} \geq -M} \right).$$

The interlacing formula of Warren (see equation (2.3)) thus gives

$$1_E(\bar{z}) = 1_{M \geq z_1^{(1)} \geq -M} \left(\prod_{j=2}^n \delta_{c_j}(z_j^{(1)}) \right) \left(\prod_{r=1}^{n-1} \det [1_{M \geq z_k^{(r+1)} > z_j^{(r)} \geq c_n}]_{j,k=1}^n \right) \left(\prod_{i=1}^n 1_{M \geq z_i^{(n)} \geq -M} \right).$$

The required result in this case follows from equation (2.4). The result when $\bar{z} \in (\mathfrak{S}_n \mathcal{C}_n)^n$ follows since the expressions given in equations (2.4) and (2.5) are invariant under permutations. Finally, the result is trivially true when $\bar{z} \in (\mathbb{R}^n)^n \setminus (\mathfrak{S}_n \mathcal{C}_n)^n$, since both expressions are identically 0. \square

Equation (2.5) gives

$$Z = \frac{1}{(n!)^n} \int_{(\mathbb{R}^n)^n} \prod_{r=0}^n \det[\phi_{r,r+1}(z_j^{(r)}, z_k^{(r+1)})]_{j,k=1}^n dz^{(n)} dz^{(n-1)} \dots dz^{(1)}.$$

The Cauchy-Binet identity (Proposition 2.10 of Johansson, [18]) then gives $Z = \det A$, where $A \in \mathbb{C}^{n \times n}$ is given by $A_{ij} := \phi_{0,n+1}(i, j)$ for all $i, j \in \{1, \dots, n\}$, and $\phi_{0,s} : \{1, \dots, n\} \times \mathbb{R} \rightarrow \mathbb{R}$, $\phi_{r,s} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi_{r,n+1} : \mathbb{R} \times \{1, \dots, n\} \rightarrow \mathbb{R}$ are defined by

$$\phi_{r,s}(u, v) := 1_{s>r} \int_{\mathbb{R}} dz_1 \dots \int_{\mathbb{R}} dz_{s-r-1} \phi_{r,r+1}(u, z_1) \phi_{r+1,r+2}(z_1, z_2) \dots \phi_{s-1,s}(z_{s-r-1}, v),$$

for all $r, s \in \{1, \dots, n\}$. Therefore

$$(2.6) \quad \phi_{r,s}(u, v) = \frac{(v-u)^{s-r-1}}{(s-r-1)!} 1_{M \geq v > u \geq c_n} 1_{s>r},$$

$$(2.7) \quad \phi_{0,s}(i, v) = \frac{(v-c_i)^{s-2+1_{i=1}}}{(s-2+1_{i=1})!} 1_{M \geq v > c_i},$$

$$(2.8) \quad \phi_{r,n+1}(u, j) = \int_{-M}^M dz \phi_j(z) \frac{(z-u)^{n-r-1}}{(n-r-1)!} 1_{z>u \geq c_n} 1_{r \leq n-1} + \phi_j(u) 1_{M \geq u \geq -M} 1_{r=n},$$

$$(2.9) \quad A_{ij} = \int_{-M}^M dz \phi_j(z) \frac{(z-c_i)^{n-2+1_{i=1}}}{(n-2+1_{i=1})!},$$

for all $r, s, i, j \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$.

Proposition 2.3. *Letting $\Phi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ be that given in equation (2.2), and $M > 0$ be that given in equation (2.4), define $B_n^{(M)} := \int_{[-M, M]^n} dz \Phi_n(z)$. Then $B_n^{(M)} > 0$, and the random point field $((\mathbb{R}^n)^n, \xi_n)$ is determinantal with correlation kernel $J_n : (\{1, \dots, n\} \times \mathbb{R})^2 \rightarrow \mathbb{R}$, which satisfies*

$$\begin{aligned} J_n((r, u), (s, v)) &= \frac{1}{B_n^{(M)}} \int_{\mathfrak{S}_n \mathcal{C}_n^{(M)}} dz \Phi_n(z) \sum_{j=1}^n 1_{v \leq u < z_j} \frac{(z_j - u)^{n-r-1}}{(n-r-1)!} \frac{\partial^{n-s}}{\partial v^{n-s}} \prod_{i \neq j} \left(\frac{v - z_i}{z_j - z_i} \right) \\ &\quad - \frac{1}{B_n^{(M)}} \int_{\mathfrak{S}_n \mathcal{C}_n^{(M)}} dz \Phi_n(z) \sum_{j=1}^n 1_{v > u \geq z_j} \frac{(z_j - u)^{n-r-1}}{(n-r-1)!} \frac{\partial^{n-s}}{\partial v^{n-s}} \prod_{i \neq j} \left(\frac{v - z_i}{z_j - z_i} \right), \end{aligned}$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in (-M, M)$, where $\mathcal{C}_n^{(M)} := \mathcal{C}_n \cap [-M, M]^n$.

PROOF. The fact that $((\mathbb{R}^n)^n, \xi_n)$ is determinantal follows from Lemma 2.2 and Proposition 2.13 of Johansson, [18]. A correlation kernel is given by

$$(2.10) \quad J_n((r, u), (s, v)) = -\phi_{r,s}(u, v) + \tilde{J}_n((r, u), (s, v)),$$

for all $r, s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$, where

$$\tilde{J}_n((r, u), (s, v)) = \sum_{i,j=1}^n (-1)^{i+j} \phi_{0,s}(i, v) \frac{\det A(i, j)}{\det A} \phi_{r,n+1}(u, j),$$

and $A(i, j) \in \mathbb{C}^{(n-1) \times (n-1)}$ is the sub-matrix of A obtained by removing row i and column j .

Fix $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in (-M, M)$. First note equation (2.6) gives

$$(2.11) \quad \phi_{r,s}(u, v) = 1_{v>u} \frac{\partial^{n-s}}{\partial v^{n-s}} \frac{(v-u)^{n-r-1}}{(n-r-1)!} = 1_{v>u} \frac{\partial^{n-s}}{\partial v^{n-s}} \sum_{j=1}^n \frac{(z_j-u)^{n-r-1}}{(n-r-1)!} \prod_{i \neq j} \left(\frac{v-z_i}{z_j-z_i} \right),$$

for any $z \in \mathfrak{S}_n \mathcal{C}_n$, where the last step follows from Lagrange interpolation. Also equations (2.7) and (2.8) give

$$(2.12) \quad \tilde{J}_n((r, u), (s, v)) = \frac{\partial^{n-s}}{\partial v^{n-s}} \sum_{j=1}^n \frac{\det A^{(j,v)}}{\det A} \int_{-M}^M dz_j \phi_j(z_j) \frac{(z_j-u)^{n-r-1}}{(n-r-1)!} 1_{z_j>u},$$

where $A^{(j,v)} \in \mathbb{C}^{n \times n}$ is A with column j replaced by $(\phi_{0,n}(1, v), \dots, \phi_{0,n}(n, v))^T$. This can be verified by taking a cofactor expansion of $\det A^{(j,v)}$ along column j . Moreover equation (2.9) gives

$$\det A = \int_{[-M, M]^n} dz \sum_{l_1=0}^{n-1} \sum_{l_2, \dots, l_n=0}^{n-2} \left(\prod_{k=1}^n \frac{\phi_k(z_k) (-c_k)^{n-2+1_{k=1}-l_k}}{(l_k)! (n-2+1_{k=1}-l_k)!} \right) \det [(z_j)^{l_i}]_{i,j=1}^n.$$

The only non-zero terms in the above sum are those for which l_1, \dots, l_n are distinct. Equation (1.7) then gives

$$\det A = \frac{1}{(n-1)!} \left(\prod_{k=0}^{n-2} \frac{1}{k!} \right)^2 \Delta_{n-1}(c_2, \dots, c_n) B_n^{(M)},$$

where $B_n^{(M)}$ is defined in the statement of the Proposition. Therefore $B_n^{(M)} > 0$, since $(c_1, \dots, c_n) \in \mathcal{C}_n$ and $\det A = Z$, where $Z > 0$ is the normalisation constant in equation (2.4). Similarly

$$\det A^{(j,v)} = \frac{1}{(n-1)!} \left(\prod_{k=0}^{n-2} \frac{1}{k!} \right)^2 \Delta_{n-1}(c_2, \dots, c_n) \int_{[-M, M]^{n-1}} \left(\prod_{k \neq j} \phi_k(z_k) dz_k \right) \Delta_n(z^{(j,v)}),$$

for all $j \in \{1, \dots, n\}$, where $z^{(j,v)} := (z_1, \dots, z_{j-1}, v, z_{j+1}, \dots, z_n)$. Equations (1.7), (2.2) and (2.12) thus give

$$\tilde{J}_n((r, u), (s, v)) = \frac{1}{B_n^{(M)}} \frac{\partial^{n-s}}{\partial v^{n-s}} \int_{\mathfrak{S}_n \mathcal{C}_n^{(M)}} dz \Phi_n(z) \sum_{j; z_j > u} \frac{(z_j-u)^{n-r-1}}{(n-r-1)!} \prod_{i \neq j} \left(\frac{v-z_i}{z_j-z_i} \right).$$

Equations (2.10) and (2.11) then give the required result. \square

We are now in a position to give a proof of Theorem 2.1:

PROOF OF THEOREM 2.1. For each $(c_1, \dots, c_n) \in \mathcal{C}_n$ with $c_1 = -M < 0$, using superscripts to emphasise the dependence on (c_1, \dots, c_n) , Proposition 2.3 implies that $((\mathbb{R}^n)^n, \xi_n^{(c_1, \dots, c_n)})$ is a determinantal random point field with correlation kernel $J_n^{(c_1, \dots, c_n)} : (\{1, \dots, n\} \times \mathbb{R})^2 \rightarrow \mathbb{R}$. When restricted to the domain $(\{1, \dots, n-1\} \times (-M, M)) \times (\{1, \dots, n\} \times (-M, M))$, this kernel depends on M and does not depend on c_2, \dots, c_n . Also it follows from equations (2.1) and (2.4) that $\xi_n^{(c_1, \dots, c_n)}$ induces the probability measure on GT_n given by

$$\nu_n^{(M)}[A] := \frac{\nu_n[A \cap (\mathcal{C}_1^{(M)} \times \dots \times \mathcal{C}_n^{(M)})]}{\nu_n[\text{GT}_n \cap (\mathcal{C}_1^{(M)} \times \dots \times \mathcal{C}_n^{(M)})]},$$

for all $A \subset \text{GT}_n$ measurable, where $\mathcal{C}_r^{(M)} = \mathcal{C}_r \cap [-M, M]^r$ for all $r \in \{1, \dots, n\}$. Therefore $(\text{GT}_n, \nu_n^{(M)})$ is a determinantal random point field with correlation kernel $K_n^{(M)} : (\{1, \dots, n\} \times (-M, M))^2 \rightarrow \mathbb{R}$ which

satisfies $K_n^{(M)} = J_n^{(c_1, \dots, c_n)}$ in the domain $(\{1, \dots, n-1\} \times (-M, M)) \times (\{1, \dots, n\} \times (-M, M))$. The required result follows by letting $M \rightarrow \infty$. \square

We finish this section by obtaining useful contour integral expressions for the kernel in Theorem 2.1:

Proposition 2.4. *For all $r \in \{1, \dots, n-2\}$, $s \in \{1, \dots, n\}$, and $u, v \in \mathbb{R}$,*

$$K_n((r, u), (s, v)) = \frac{1}{(2\pi)^2} \frac{(n-s)!}{(n-r-1)!} \frac{1}{B_n} \int_{\mathfrak{S}_n \mathcal{C}_n} dy \Phi_n(y) \times \\ \times \int_{\gamma(u, v, y)} dw \int_{\Gamma(u, v, y)} dz \frac{(z-u)^{n-r-1}}{(w-v)^{n-s+1}} \frac{1}{w-z} \prod_{i=1}^n \left(\frac{w-y_i}{z-y_i} \right).$$

Here $\gamma(u, v, y)$ is a counter-clockwise simple closed contour around v . Whenever $v \leq u$, $\Gamma(u, v, y)$ is a clockwise simple closed contour which passes through u , contains $\{y_j : y_j > u\}$ and does not contain $\{y_j : y_j < u\}$. Whenever $v > u$, $\Gamma(u, v, y)$ is a counter-clockwise simple closed contour which passes through u , contains $\{y_j : y_j < u\}$ and does not contain $\{y_j : y_j > u\}$. Finally the contours do not intersect. This holds with the understanding that $(z-u)^{n-r-1} \prod_i (\frac{1}{z-y_i}) = (z-u)^{n-r-2} \prod_{i \neq k} (\frac{1}{z-y_i})$ whenever $u = y_k$ for some $k \in \{1, \dots, n\}$.

Also for all $r \in \{1, \dots, n-1\}$ and $u, v \in \mathbb{R}$,

$$K_n((r, u), (r, v)) = \frac{1}{(2\pi)^2} \frac{1}{B_n} \int_{\mathfrak{S}_n \mathcal{C}_n} dy \Phi_n(y) \times \\ \times \int_{\gamma} dw \int_{\Gamma(u, v, y)} dz \left(\frac{(z+v-u)^{n-r} - z^{n-r}}{(v-u)w^{n-r+1}} \right) \sum_{j=1}^n \frac{v-y_j}{(z+v-y_j)^2} \prod_{i \neq j} \left(\frac{w+v-y_i}{z+v-y_i} \right).$$

Here γ is a counter-clockwise simple closed contour around 0. Whenever $v \leq u$, $\Gamma(u, v, y)$ is a clockwise simple closed contour in $\mathbb{C} \setminus \{y_1 - v, \dots, y_n - v\}$ which contains $\{y_j - v : y_j > u\}$ and does not contain $\{y_j - v : y_j \leq u\}$. Whenever $v > u$, $\Gamma(u, v, y)$ is a counter-clockwise simple closed contour in $\mathbb{C} \setminus \{y_1 - v, \dots, y_n - v\}$ which contains $\{y_j - v : y_j \leq u\}$ and does not contain $\{y_j - v : y_j > u\}$. This holds with the understanding that $\frac{(z+v-u)^{n-r} - z^{n-r}}{v-u} = (n-r)z^{n-r-1}$ whenever $u = v$.

PROOF. For all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$, Theorem (1.6) gives

$$(2.13) \quad K_n((r, u), (s, v)) = \frac{1}{(2\pi)^2} \frac{(n-s)!}{(n-r-1)!} \frac{1}{B_n} \int_{\mathfrak{S}_n \mathcal{C}_n} dy \Phi_n(y) G_{r,s}^{(u,v)}(y),$$

where $G_{r,s}^{(u,v)} : \mathfrak{S}_n \mathcal{C}_n \rightarrow \mathbb{R}$ is given by

$$G_{r,s}^{(u,v)}(y) := (2\pi)^2 \sum_{j=1}^n 1_{v \leq u < y_j} (y_j - u)^{n-r-1} e_{s-1} \left(v - y_1, \dots, \widehat{v - y_j}, \dots, v - y_n \right) \prod_{i \neq j} \left(\frac{1}{y_j - y_i} \right) \\ (2.14) \quad - (2\pi)^2 \sum_{j=1}^n 1_{v > u \geq y_j} (y_j - u)^{n-r-1} e_{s-1} \left(v - y_1, \dots, \widehat{v - y_j}, \dots, v - y_n \right) \prod_{i \neq j} \left(\frac{1}{y_j - y_i} \right),$$

for all $y \in \mathfrak{S}_n \mathcal{C}_n$. Here e_{s-1} is the elementary symmetric polynomial of degree $s-1$. Then, whenever $r \leq n-2$, the residue Theorem gives the first part of the result. To see the second part note that the residue Theorem alternatively gives

$$G_{r,r}^{(u,v)}(y) = \int_{\gamma(u, v, y)} dw \int_{\Gamma(u, v, y)} dz \frac{(z+v-u)^{n-r-1}}{w^{n-r+1}} \frac{1}{w-z} \prod_{i=1}^n \left(\frac{w+v-y_i}{z+v-y_i} \right),$$

for all $r \in \{1, \dots, n-1\}$, $u, v \in \mathbb{R}$, and $y \in \mathfrak{S}_n \mathcal{C}_n$, where we choose the contours so that they do not intersect, $\gamma(u, v, y)$ is not in the interior of $\Gamma(u, v, y)$, $\gamma(u, v, y)$ is a counter-clockwise simple closed contour around 0, and $\Gamma(u, v, y)$ is chosen as in the second part of the result. Fixing $r \in \{1, \dots, n-1\}$, $u, v \in \mathbb{R}$, and $y \in \mathfrak{S}_n \mathcal{C}_n$, write

$$(2.15) \quad G_{r,r}^{(u,v)}(y) = \sum_{m=1}^{n-r} \binom{n-r}{m-1} (v-u)^{m-1} F_m,$$

where

$$F_m := \int_{\gamma(u,v,y)} dw \int_{\Gamma(u,v,y)} dz \frac{z^{n-r-m}}{w^{n-r+1}} \frac{1}{w-z} \prod_{i=1}^n \left(\frac{w+v-y_i}{z+v-y_i} \right).$$

Note, for all $b \in \mathbb{R}$ sufficiently close to 1, the residue Theorem implies that $\gamma(u, v, y)$ and $\Gamma(u, v, y)$ can be replaced by $b \gamma(u, v, y)$ and $b \Gamma(u, v, y)$ respectively, and so

$$F_m = b^{-m} \int_{\gamma(u,v,y)} dw \int_{\Gamma(u,v,y)} dz \frac{z^{n-r-m}}{w^{n-r+1}} \frac{1}{w-z} \prod_{i=1}^n \left(\frac{bw+v-y_i}{bz+v-y_i} \right).$$

Differentiate both sides with respect to b and set $b = 1$ to get

$$F_m = \frac{1}{m} \int_{\gamma(u,v,y)} dw \int_{\Gamma(u,v,y)} dz \frac{z^{n-r-m}}{w^{n-r+1}} \sum_{j=1}^n \frac{v-y_j}{(z+v-y_j)^2} \prod_{i \neq j} \left(\frac{w+v-y_i}{z+v-y_i} \right).$$

The residue Theorem implies that $\gamma(u, v, y)$ can be replaced by any counter-clockwise simple closed contour around 0, γ . Equation (2.15) then gives

$$G_{r,r}^{(u,v)}(y) = \sum_{m=1}^{n-r} \binom{n-r}{m} \frac{(v-u)^{m-1}}{n-r} \int_{\gamma} dw \int_{\Gamma(u,v,y)} dz \frac{z^{n-r-m}}{w^{n-r+1}} \sum_{j=1}^n \frac{v-y_j}{(z+v-y_j)^2} \prod_{i \neq j} \left(\frac{w+v-y_i}{z+v-y_i} \right).$$

This holds for all $r \in \{1, \dots, n-1\}$, $u, v \in \mathbb{R}$, and $y \in \mathfrak{S}_n \mathcal{C}_n$. Equation (2.13) gives the required result. \square

3. Proof of Theorem 1.6

In this section we prove Theorem 1.6. Fix $a, b \in \mathbb{R}$ with $a < b$. For each $n \in \mathbb{N}$, choose $q_n \in \{1, \dots, n\}$ and $x^{(n)} \in \mathcal{C}_n \cap [a, b]^n$ as in sections 1.3 and 1.4, and equip GT_n with the measure given in equation (1.12). This satisfies equation (2.1) with $\phi_i = \delta_{x_i^{(n)}}$ for all $i \in \{1, \dots, n\}$. Let $K_n : (\{1, \dots, n\} \times [a, b])^2 \rightarrow \mathbb{C}$ be the associated correlation kernel given equation in (1.13).

Assume hypothesis 1.1. Fix $c \in A_\alpha$ and $U, V \subset \mathbb{R}$ compact, where $A_\alpha \subset (a, b)$ is given in equation (1.14). Proposition 2.4 gives

$$(3.1) \quad \frac{4\pi^2}{n} K_n \left(\left(q_n, c + \frac{u}{n} \right), \left(q_n, c + \frac{v}{n} \right) \right) = \int_{\gamma_n} dw \int_{\Gamma_n} dz \left(\frac{(z + \frac{v-u}{n})^{n-q_n} - z^{n-q_n}}{(v-u)w^{n-q_n+1}} \right) \sum_{j=1}^n \frac{c + \frac{v}{n} - x_j^{(n)}}{(z + c + \frac{v}{n} - x_j^{(n)})^2} \prod_{i \neq j} \left(\frac{w + c + \frac{v}{n} - x_i^{(n)}}{z + c + \frac{v}{n} - x_i^{(n)}} \right),$$

for all n sufficiently large, $u \in U$ and $v \in V$, where γ_n is a counter-clockwise simple closed contour around 0, and Γ_n is a simple closed contour in $\mathbb{C} \setminus \{x_1^{(n)} - \frac{v}{n} - c, \dots, x_n^{(n)} - \frac{v}{n} - c\}$ which satisfies

- Whenever $v \leq u$, Γ_n is clockwise, contains $\{x_j^{(n)} - \frac{v}{n} - c : x_j^{(n)} > \frac{u}{n} + c\}$ and does not contain $\{x_j^{(n)} - \frac{v}{n} - c : x_j^{(n)} \leq \frac{u}{n} + c\}$.

- Whenever $v > u$, Γ_n is counter-clockwise, contains $\{x_j^{(n)} - \frac{v}{n} - c : x_j^{(n)} \leq \frac{u}{n} + c\}$ and does not contain $\{x_j^{(n)} - \frac{v}{n} - c : x_j^{(n)} > \frac{u}{n} + c\}$.

We examine the asymptotics of this kernel via saddle point analysis. First note, for all n sufficiently large, $u \in U$, $v \in V$, and $z, w \in \mathbb{C} \setminus \mathbb{R}$, the integrand can be rewritten as

$$(3.2) \quad n \left(\frac{(1 + \frac{v-u}{nz})^{n-q_n} - 1}{v-u} \right) g_{n,v}(w, z) e^{n(h_{n,v}(w) - h_{n,v}(z))},$$

where, using the principal value of the logarithm, $h_{n,v} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ and $g_{n,v} : (\mathbb{C} \setminus \mathbb{R})^2 \rightarrow \mathbb{C}$ are given by

$$(3.3) \quad h_{n,v}(w) := \int \log(w + c - x) \mu_{n,v}[dx] - \frac{n - q_n}{n} \log(w),$$

$$(3.4) \quad g_{n,v}(w, z) := \begin{cases} \frac{h'_{n,v}(w) - h'_{n,v}(z)}{w - z} + \frac{h'_{n,v}(w)}{w}; & w \neq z, \\ h''_{n,v}(w) + \frac{h'_{n,v}(w)}{w}; & w = z, \end{cases}$$

and $\mu_{n,v}$ is the empirical probability measure

$$(3.5) \quad \mu_{n,v} := \frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n)} - \frac{v}{n}}.$$

The following Lemma proves the existence of appropriate saddle points of $h_{n,v}$ for the analysis, and the first part of Theorem 1.6.

Lemma 3.1. *Define $h : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ by*

$$(3.6) \quad h(w) := \int \log(w + c - x) \mu[dx] - (1 - \alpha) \log(w),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Then there exists a $w_0 \in \mathbb{C}$ with $\text{Im}(w_0) > 0$ and $\{w \in \mathbb{C} \setminus \mathbb{R} : h'(w) = 0\} = \{w_0, \overline{w_0}\}$. Also $h''(w_0) \neq 0$. Moreover, given $v \in V$ and n sufficiently large, there exists a $w_{n,v} \in \mathbb{C}$ with $\text{Im}(w_{n,v}) > 0$ and $\{w \in \mathbb{C} \setminus \mathbb{R} : h'_{n,v}(w) = 0\} = \{w_{n,v}, \overline{w_{n,v}}\}$. Finally

$$\lim_{n \rightarrow \infty} \sup_{v \in V} |w_{n,v} - w_0| = 0.$$

PROOF. Since roots of h' and $h'_{n,v}$ occur in complex conjugate pairs, we shall restrict our attention to $\{w \in \mathbb{C} : \text{Im}(w) > 0\}$. Equations (3.3) and (3.5) give

$$nw \prod_{i=1}^n (w + c + \frac{v}{n} - x_j^{(n)}) h'_{n,v}(w) = w \sum_{j=1}^n \prod_{i \neq j} (w + c + \frac{v}{n} - x_i^{(n)}) - (n - q_n) \prod_{i=1}^n (w + c + \frac{v}{n} - x_j^{(n)}),$$

for all n sufficiently large, $v \in V$ and $w \in \mathbb{C}$ with $\text{Im}(w) > 0$. The right hand side, a polynomial of degree n with real coefficients, has at least $n - 2$ roots in \mathbb{R} . Thus $h'_{n,v}$ has at most one root (counting multiplicities) in $\{w \in \mathbb{C} : \text{Im}(w) > 0\}$.

Since $c \in A_\alpha$, equations (1.14) and (3.6) imply that h' has at least one root in $\{w \in \mathbb{C} : \text{Im}(w) > 0\}$. Denoting this by w_0 , we now show that, for any $\epsilon \in (0, \text{Im}(w_0))$ and $j \geq 0$

$$(3.7) \quad \lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{w \in B(w_0, \epsilon)} |h_{n,v}^{(j)}(w) - h^{(j)}(w)| = 0.$$

We use the method of contradictions to prove the result for $j = 0$. Assume that this does not hold for some $\epsilon \in (0, \text{Im}(w_0))$. Thus there exists some $\xi > 0$ for which, for all $n \geq 1$, there exists some $m_n \geq n$ and

$z_n \in \bar{B}(w_0, \epsilon)$ with $\xi \leq \sup_{v \in V} |h_{m_n, v}(z_n) - h(z_n)|$. Also the Bolzano-Weierstrass Theorem implies that we can choose $\{z_n\}_{n \geq 1}$ to be convergent. Denoting the limit by z_0 ,

$$\xi \leq \sup_{v \in V} |h_{m_n, v}(z_n) - h_{m_n, v}(z_0)| + \sup_{v \in V} |h_{m_n, v}(z_0) - h(z_0)| + |h(z_0) - h(z_n)|,$$

for all n sufficiently large. Finally note equation (3.3) gives $\sup\{|h'_{m_n, v}(w)| : v \in V \text{ and } w \in \bar{B}(w_0, \epsilon)\} \leq 2(|\text{Im}(w_0)| - \epsilon)^{-1}$ for all n sufficiently large, and so

$$\sup_{v \in V} |h_{m_n, v}(z_0) - h(z_0)| \geq \frac{\xi}{2}$$

for all n sufficiently large. However, since $\frac{q_n}{n} \rightarrow \alpha$ and $\mu_{n,0} \rightarrow \mu$ weakly (see equation (3.5) and hypothesis 1.1), equations (3.3) and (3.6) imply that this is false. Thus equation (3.7) is true when $j = 0$. The result for $j \geq 1$ follows from Cauchy estimates.

Now, since h' is a non-constant analytic function on $\{w \in \mathbb{C} : \text{Im}(w) > 0\}$ with $h'(w_0) = 0$, then $h'(w) \neq 0$ for all $w \in B(w_0, \epsilon) \setminus \{w_0\}$ and all $\epsilon > 0$ sufficiently small. Thus, letting $\partial B(w_0, \epsilon)$ be the boundary of $B(w_0, \epsilon)$, the Bolzano-Weierstrass Theorem gives

$$\inf_{w \in \partial B(w_0, \epsilon)} |h'(w)| > 0,$$

for all $\epsilon > 0$ sufficiently small. It thus follows from equation (3.7) and Rouché's Theorem that there exists a function $N : \mathbb{R}_+ \rightarrow \mathbb{N}$ for which h' and $h'_{n,v}$ have the same number of roots in $B(w_0, \epsilon)$ (counting multiplicities) for all $\epsilon > 0$ sufficiently small, $v \in V$ and $n \geq N(\epsilon)$. Since this can be done for any $\epsilon > 0$ sufficiently small, the required results follow from the above observation that $h'_{n,v}$ has at most one root (counting multiplicities) in $\{w \in \mathbb{C} : \text{Im}(w) > 0\}$. \square

For notational purposes set $w_0^+ := w_0$, $w_0^- := \overline{w_0}$, $w_{n,v}^+ := w_{n,v}$ and $w_{n,v}^- := \overline{w_{n,v}}$.

Remark 3.1. Some useful observations: Equations (3.3) and (3.6), and Lemma 3.1, give

$$\int \frac{w_{n,v}^\pm}{w_{n,v}^\pm + c - x} \mu_{n,v}[dx] = \frac{n - q_n}{n} \quad \text{and} \quad \int \frac{w_0^\pm}{w_0^\pm + c - x} \mu[dx] = 1 - \alpha,$$

for all n sufficiently large and $v \in V$. Comparing real and imaginary parts,

$$(3.8) \quad \int \frac{c - x}{|w_{n,v}^\pm + c - x|^2} \mu_{n,v}[dx] = 0, \quad \int \frac{c - x}{|w_0^\pm + c - x|^2} \mu[dx] = 0,$$

$$(3.9) \quad \int \frac{|w_{n,v}^\pm|^2}{|w_{n,v}^\pm + c - x|^2} \mu_{n,v}[dx] = \frac{n - q_n}{n}, \quad \int \frac{|w_0^\pm|^2}{|w_0^\pm + c - x|^2} \mu[dx] = 1 - \alpha,$$

for all n sufficiently large and $v \in V$.

We now fix the contours γ_n and Γ_n of equation (3.1). We define them to pass through $w_{n,v}^\pm$ so that a saddle point asymptotic analysis can be performed, i.e., the integral can be estimated using small sections of the contours around $w_{n,v}^\pm$. Equation (3.2) implies that we need to choose them so that $w \mapsto |e^{h_n(w)}|$ and $z \mapsto |e^{-h_n(z)}|$, for all w on γ_n and z on Γ_n , are both maximised at $w_{n,v}^\pm$.

Lemma 3.1 and equation (3.7) show that $h''(w_0^+) \neq 0$ and

$$(3.10) \quad \lim_{n \rightarrow \infty} \sup_{v \in V} |h''_{n,v}(w_{n,v}^\pm) - h''(w_0^\pm)| = 0.$$

Thus we can define $\theta_{n,v} := -\frac{1}{2} \text{Arg}(h''_{n,v}(w_{n,v}^\pm))$ for all n sufficiently large and $v \in V$, where $\text{Arg}(w) \in (-\pi, \pi]$ is the argument of w . Then, fixing $\delta \in (\frac{1}{3}, \frac{1}{2})$, and defining $\varepsilon_0 := (-1)^{1_{C_0 \geq 0}}$ where $C_0 \in \mathbb{R}$ is

given in equation (3.27), define

$$(3.11) \quad \begin{aligned} \gamma_{n,1}^+(s) &:= |w_{n,v}^+ - i\varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}| e^{is}, & s \in [0, \frac{1}{n}), \\ \gamma_{n,2}^+(s) &:= |w_{n,v}^+ - i\varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}| e^{is}, & s \in [\frac{1}{n}, \text{Arg}(w_{n,v}^+ - i\varepsilon_0 n^{-\delta} e^{i\theta_{n,v}})), \\ \gamma_{n,3}^+(s) &:= w_{n,v}^+ + i\varepsilon_0 n^{-\delta} e^{i\theta_{n,v}} s, & s \in [-1, 1], \\ \gamma_{n,4}^+(s) &:= |w_{n,v}^+ + i\varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}| e^{is}, & s \in (\text{Arg}(w_{n,v}^+ + i\varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}), \pi - \frac{1}{n}], \\ \gamma_{n,5}^+(s) &:= |w_{n,v}^+ + i\varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}| e^{is}, & s \in (\pi - \frac{1}{n}, \pi]. \end{aligned}$$

Take $\gamma_n := \sum_{j=1}^5 (\gamma_{n,j}^+ + \gamma_{n,j}^-)$, where $\gamma_{n,j}^-$ is the contour with counter-clockwise orientation obtained by reflecting $\gamma_{n,j}^+$ through the real line. Also define $\Gamma : \{z \in \mathbb{C} : \text{Im}(z) > 0\} \times (0, \infty) \rightarrow \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ by

$$(3.12) \quad \Gamma(w, s) := s|w| \exp \left(i \arccos \left(\cos(\text{Arg}(w)) \frac{2s \log(s)}{s^2 - 1} \right) \right),$$

for all $w \in \mathbb{C}$ with $\text{Im}(w) > 0$ and $s > 0$, where $\arccos : (-1, 1) \rightarrow (0, \pi)$ is the principal value of the inverse cosine function. Note, for any fixed $w \in \mathbb{C}$ with $\text{Im}(w) > 0$, the contour $\Gamma(w, \cdot) : (0, \infty) \rightarrow \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ is well-defined and continuous and satisfies $\Gamma(w, 1) = w$ and $\lim_{s \rightarrow 0} \text{Arg}(\Gamma(w, s)) = \lim_{s \rightarrow \infty} \text{Arg}(\Gamma(w, s)) = \frac{\pi}{2}$. Then, for all n sufficiently large, $u \in U$ and $v \in V$, define

$$(3.13) \quad \begin{aligned} \Gamma_{n,1}^+(s) &:= (1-s)y_{n,u,v} + s \Gamma(w_{n,v}^+ - \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}, t_0), & s \in [0, 1), \\ \Gamma_{n,2}^+(s) &:= \Gamma(w_{n,v}^+ - \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}, s), & s \in [t_0, 1), \\ \Gamma_{n,3}^+(s) &:= w_{n,v}^+ + \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}} s, & s \in [-1, 1], \\ \Gamma_{n,4}^+(s) &:= \Gamma(w_{n,v}^+ + \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}, s), & s \in (1, T_0), \end{aligned}$$

where $t_0 \in (0, 1)$, $T_0 > 1$, and $y_{n,u,v} \in (x_j^{(n)} - \frac{v}{n} - c, x_{j-1}^{(n)} - \frac{v}{n} - c)$ for that value of j which satisfies $c + \frac{u}{n} \in [x_j^{(n)}, x_{j-1}^{(n)})$. These quantities will be fixed in Lemma 3.2. Finally define $\Gamma_{n,5}^+$ to be the contour that spans the segment of the circle centered at the origin, starting at $\Gamma(w_{n,v}^+ + \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}, T_0)$, ending on the real line, with clockwise orientation when $v \leq u$ and counter-clockwise orientation when $v > u$. Take $\Gamma_n := \sum_{j=1}^5 (\Gamma_{n,j}^+ + \Gamma_{n,j}^-)$, where $\Gamma_{n,j}^-$ is the contour obtained by reflecting $\Gamma_{n,j}^+$ through the real line.

Let $\gamma : [0, \pi] \rightarrow \mathbb{C}$ be the contour given by $\gamma(s) := |w_0^+| e^{is}$ for all $s \in [0, \pi]$. It follows from Lemma 3.1 and equation (3.11) that this can be regarded as the ‘limit contour’ in the upper half complex plane of $\{\gamma_n\}_{n \geq 1}$. Also equations (3.12) and (3.13) show that $\Gamma(w_0^+, \cdot) : (0, \infty) \rightarrow \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ can be regarded as the ‘limit contour’ in the upper half complex plane of $\{\Gamma_n\}_{n \geq 1}$. As we shall see in Lemmas 3.3 and 3.4, the functions given by $w \mapsto |e^{h(w)}|$ and $z \mapsto |e^{-h(z)}|$, for all w on γ and z on $\Gamma(w_0^+, \cdot)$, are both maximised at w_0^+ . Using this fact, the properties of γ_n and Γ_n that make them suitable for saddle point analysis are shown in Lemmas 3.3 and 3.4. As we shall see, for n sufficiently large, the only significant contributions come from $\gamma_{n,3}^\pm$ and $\Gamma_{n,3}^\pm$.

For all n sufficiently large, $u \in U$, $v \in V$ and $A, B \subset \{1, 2, 3, 4, 5\}$, define $\gamma_n^A := \sum_{j \in A} (\gamma_{n,j}^+ + \gamma_{n,j}^-)$, $\Gamma_n^B := \sum_{j \in B} (\Gamma_{n,j}^+ + \Gamma_{n,j}^-)$, and

$$(3.14) \quad K_{n,u,v}^{A,B} := \int_{\gamma_n^A} dw \int_{\Gamma_n^B} dz \left(\frac{(z + \frac{v-u}{n})^{n-q_n} - z^{n-q_n}}{(v-u)w^{n-q_n+1}} \right) \sum_{j=1}^n \frac{c + \frac{v}{n} - x_j^{(n)}}{(z + c + \frac{v}{n} - x_j^{(n)})^2} \prod_{i \neq j} \left(\frac{w + c + \frac{v}{n} - x_i^{(n)}}{z + c + \frac{v}{n} - x_i^{(n)}} \right).$$

For all n sufficiently large, $u \in U$ and $v \in V$, equation (3.1) then gives

$$\begin{aligned} & \frac{4\pi^2}{n} K_n \left(\left(q_n, c + \frac{u}{n} \right), \left(q_n, c + \frac{v}{n} \right) \right) \\ &= K_{n,u,v}^{3,3} + K_{n,u,v}^{\{1,2,4,5\},3} + K_{n,u,v}^{\{1,2,3,4,5\},1} + K_{n,u,v}^{\{1,2,3,4,5\},\{2,4\}} + K_{n,u,v}^{\{1,2,3,4,5\},5}. \end{aligned}$$

Lemmas 3.2, 3.3 and 3.4 imply that there exists constants $c > 0$ and $C > 1$ for which

$$\begin{aligned} \sup_{u \in U, v \in V} \left| K_{n,u,v}^{\{1,2,3,4,5\},1} + K_{n,u,v}^{\{1,2,3,4,5\},5} \right| &\leq C^{-n}, \\ \sup_{u \in U, v \in V} \left| K_{n,u,v}^{\{1,2,4,5\},3} + K_{n,u,v}^{\{1,2,3,4,5\},\{2,4\}} \right| &\leq n^3 e^{-cn^{1-2\delta}}, \end{aligned}$$

for all n sufficiently large. We now give a proof of Theorem 1.6:

PROOF OF THEOREM 1.6: The first part of this Theorem was shown in Lemma 3.1 (note, in Lemma 3.1 we denoted $w_{\alpha,c}$ by w_0 for simplicity of notation). It remains to show the asymptotic limit. Since $\delta \in (\frac{1}{3}, \frac{1}{2})$, the above equation and bounds imply that the result follows if and only if

$$(3.15) \quad \lim_{n \rightarrow \infty} \sup_{u \in U, v \in V} \left| K_{n,u,v}^{3,3} - 4\pi(C_{\alpha,c})^{-\rho_{\alpha}(c)(v-u)} \frac{\sin(\pi\rho_{\alpha}(c)(v-u))}{v-u} \right| = 0,$$

where $\rho_{\alpha}(c) = -\frac{1-\alpha}{\pi} \text{Im}(w_0^{-1})$ and $C_{\alpha,c} = \exp\left(\pi \frac{\text{Re}(w_0^{-1})}{\text{Im}(w_0^{-1})}\right)$.

Equations (3.2) and (3.14) give

$$(3.16) \quad K_{n,u,v}^{3,3} = K_{n,u,v}^{++} + K_{n,u,v}^{--} + K_{n,u,v}^{+-} + K_{n,u,v}^{-+},$$

for all n sufficiently large, $u \in U$ and $v \in V$, where for $b, d \in \{-, +\}$,

$$K_{n,u,v}^{bd} := n \int_{\gamma_{n,3}^d} dw \int_{\Gamma_{n,3}^b} dz \left(\frac{(1 + \frac{v-u}{nz})^{n-q_n} - 1}{v-u} \right) g_{n,v}(w, z) e^{n(h_{n,v}(w) - h_{n,v}(z))},$$

and $h_{n,v}, g_{n,v}$ are defined in equations (3.3) and (3.4). Also, recalling that $h'_{n,v}(w_{n,v}^{\pm}) = 0$ (see Lemma 3.1), equations (3.7), (3.11) and (3.13) and Taylor expansions give

$$(3.17) \quad \begin{aligned} h_{n,v}(\gamma_{n,3}^b(s)) &= h_{n,v}(w_{n,v}^b) - \frac{1}{2}n^{-2\delta} |h''_{n,v}(w_{n,v}^+)| s^2 + R_{n,v}^b(s), \\ h_{n,v}(\Gamma_{n,3}^d(t)) &= h_{n,v}(w_{n,v}^d) + \frac{1}{2}n^{-2\delta} |h''_{n,v}(w_{n,v}^+)| t^2 + T_{n,v}^d(t), \end{aligned}$$

for all n sufficiently large, $v \in V$, $b, d \in \{-, +\}$ and $s, t \in [-1, 1]$, where the remainders satisfy

$$(3.18) \quad \sup_{(s,t) \in [-1,1]^2} |R_{n,v}^b(s)| + |T_{n,v}^d(t)| \leq Cn^{-3\delta},$$

for some constant $C > 0$. Therefore equations (3.11) and (3.13) give

$$K_{n,u,v}^{bd} = n^{1-2\delta} C_{n,v}^{bd} e^{n(h_{n,v}(w_{n,v}^b) - h_{n,v}(w_{n,v}^d))} \int_{-1}^1 ds \int_{-1}^1 dt A_{n,u,v}^{bd}(s, t) e^{-\frac{1}{2}n^{1-2\delta} |h''_{n,v}(w_{n,v}^+)| (s^2 + t^2)},$$

for all n sufficiently large, $u \in U$, $v \in V$ and $b, d \in \{-, +\}$, where

$$\begin{aligned} C_{n,v}^{bd} &:= \begin{cases} b i e^{-i \text{Arg}(h''_{n,v}(w_{n,v}^b))} & ; \quad b = d, \\ -b i & ; \quad b \neq d, \end{cases} \\ A_{n,u,v}^{bd}(s, t) &:= \left(\frac{(1 + \frac{v-u}{n\Gamma_{n,3}^d(t)})^{n-q_n} - 1}{v-u} \right) g_{n,v}(\gamma_{n,3}^b(s), \Gamma_{n,3}^d(t)) e^{n(R_{n,v}^b(s) - T_{n,v}^d(t))}, \end{aligned}$$

for all $s, t \in [-1, 1]$. Then, letting

$$A_{n,u,v}^{bd} := \left(\frac{e^{(1-\alpha)(v-u)(w_0^d)^{-1}} - 1}{v-u} \right) g_{n,v}(w_{n,v}^b, w_{n,v}^d),$$

and noting that $|C_{n,v}^{bd}| = |e^{n(h_{n,v}(w_{n,v}^b) - h_{n,v}(w_{n,v}^d))}| = 1$,

$$\begin{aligned} & \left| K_{n,u,v}^{bd} - \frac{2\pi C_{n,v}^{bd} A_{n,u,v}^{bd}}{|h_{n,v}''(w_{n,v}^+)|} e^{n(h_{n,v}(w_{n,v}^b) - h_{n,v}(w_{n,v}^d))} \right| \\ & \leq n^{1-2\delta} \left(\sup_{(x,y) \in [-1,1]^2} |A_{n,u,v}^{bd}(x,y) - A_{n,u,v}^{bd}| \right) \int_{-1}^1 ds \int_{-1}^1 dt e^{-\frac{1}{2}n^{1-2\delta}|h_{n,v}''(w_{n,v}^+)|(s^2+t^2)} \\ & + |A_{n,u,v}^{bd}| \left| n^{1-2\delta} \int_{-1}^1 ds \int_{-1}^1 dt e^{-\frac{1}{2}n^{1-2\delta}|h_{n,v}''(w_{n,v}^+)|(s^2+t^2)} - \frac{2\pi}{|h_{n,v}''(w_{n,v}^+)|} \right|, \end{aligned}$$

for all n sufficiently large, $u \in U$, $v \in V$ and $b, d \in \{-, +\}$ (note, it follows from equation (3.10) and Lemma 3.1 that $h_{n,v}''(w_{n,v}^+) \neq 0$ for all n sufficiently large and $v \in V$, and so these expressions are well-defined). A change of variables and equation (3.4) then gives

$$\begin{aligned} & \left| K_{n,u,v}^{bd} - b 2\pi i \left(\frac{e^{(1-\alpha)(v-u)(w_0^b)^{-1}} - 1}{v-u} \right) \delta_{bd} \right| \leq \frac{2\pi}{|h_{n,v}''(w_{n,v}^+)|} \sup_{(x,y) \in [-1,1]^2} |A_{n,u,v}^{bd}(x,y) - A_{n,u,v}^{bd}| \\ & + \delta_{bd} \left| \frac{e^{(1-\alpha)(v-u)(w_0^+)^{-1}} - 1}{v-u} \right| \left(2\pi - \int_{-|h_{n,v}''(w_{n,v}^+)|^{\frac{1}{2}}n^{\frac{1}{2}-\delta}}^{|h_{n,v}''(w_{n,v}^+)|^{\frac{1}{2}}n^{\frac{1}{2}-\delta}} ds \int_{-|h_{n,v}''(w_{n,v}^+)|^{\frac{1}{2}}n^{\frac{1}{2}-\delta}}^{|h_{n,v}''(w_{n,v}^+)|^{\frac{1}{2}}n^{\frac{1}{2}-\delta}} dt e^{-\frac{1}{2}(s^2+t^2)} \right), \end{aligned}$$

for all n sufficiently large, $u \in U$, $v \in V$ and $b, d \in \{-, +\}$, where $\delta_{bd} = 1$ if $b = d$ and $\delta_{bd} = 0$ otherwise. Recalling that $\delta \in (\frac{1}{3}, \frac{1}{2})$ and $h''(w_0) \neq 0$ (see Lemma 3.1), equations (3.4), (3.7), (3.10) and (3.18) give

$$\lim_{n \rightarrow \infty} \sup_{u \in U, v \in V} \left| K_{n,u,v}^{bd} - b 2\pi i \left(\frac{e^{(1-\alpha)(v-u)(w_0^b)^{-1}} - 1}{v-u} \right) \delta_{bd} \right| = 0,$$

for all $b, d \in \{-, +\}$. Equation (3.15) then follows from equation (3.16), as required. \square

3.1. Calculations. In this section we omit superscripts when no confusion is possible. Also we denote the range of a contour γ by γ^* . Moreover recall that $\theta_{n,v} = -\frac{1}{2} \text{Arg}(h_{n,v}''(w_{n,v}^+))$ for all n sufficiently large and $v \in V$, and $\varepsilon_0 := (-1)^{1_{C_0 \geq 0}}$, where C_0 is given in equation (3.27). Finally note equations (3.11), (3.13) and Lemma 3.1 give

$$(3.19) \quad 0 = \lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{w \in \gamma_n^*} ||w| - |w_0|| = \lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{w \in \gamma_{n,3}^*} |w - w_0| = \lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{z \in \Gamma_{n,3}^*} |z - w_0|.$$

Lemma 3.2. *There exists a constant $C > 1$, and a choice of $y_{n,u,v}$, t_0 and T_0 in equation (3.13) for which*

$$\begin{aligned} & \left| K_{n,u,v}^{\{1,2,3,4,5\},1} + K_{n,u,v}^{\{1,2,3,4,5\},5} \right| \leq C^{-n}, \\ & \left| K_{n,u,v}^{\{1,2,4,5\},3} \right| \leq \sup_{(w,z) \in (\gamma_n^{2,4})^* \times (\Gamma_n^3)^*} n^3 \left| e^{n(h_{n,v}(w) - h_{n,v}(z))} \right|, \\ & \left| K_{n,u,v}^{\{1,2,3,4,5\},\{2,4\}} \right| \leq \sup_{(w,z) \in (\gamma_n^{2,3,4})^* \times (\Gamma_n^{2,4})^*} n^3 \left| e^{n(h_{n,v}(w) - h_{n,v}(z))} \right|, \end{aligned}$$

for all n sufficiently large, $u \in U$ and $v \in V$.

PROOF. Consider $K_{n,u,v}^{\{1,2,3,4,5\},5}$. Equation (3.14) gives

$$\begin{aligned} & K_{n,u,v}^{\{1,2,3,4,5\},5} \\ & = \int_{\gamma_n} dw \int_{\Gamma_n^5} dz \left(\frac{(z + \frac{v-u}{n})^{n-q_n} - z^{n-q_n}}{(v-u)w^{n-q_n+1}} \right) \sum_{j=1}^n \frac{c + \frac{v}{n} - x_j}{(z + c + \frac{v}{n} - x_j)^2} \prod_{i \neq j} \left(\frac{w + c + \frac{v}{n} - x_i}{z + c + \frac{v}{n} - x_i} \right), \end{aligned}$$

for all n sufficiently large, $u \in U$ and $v \in V$. Recall that, for all n sufficiently large and $v \in V$, Γ_n^5 spans a segment of the circle centered at the origin, with radius $T_0|w_{n,v} + \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}|$, where $T_0 > 1$ (see equations (3.12) and (3.13)). It thus follows from equation (3.19) that there exists a constant $C > 0$, and a function $N : (0, \infty) \rightarrow \mathbb{N}$ for which

$$(3.20) \quad \left| K_{n,u,v}^{\{1,2,3,4,5\},5} \right| \leq C^n T_0^{-q_n},$$

for all T_0 sufficiently large, $n \geq N(T_0)$, $u \in U$ and $v \in V$.

Consider $K_{n,u,v}^{\{1,2,3,4,5\},\{2,4\}}$. Equation (3.14) gives

$$\begin{aligned} & K_{n,u,v}^{\{1,2,3,4,5\},\{2,4\}} \\ &= \int_{\gamma_n} dw \int_{\Gamma_n^{2,4}} dz \left(\frac{(z + \frac{v-u}{n})^{n-q_n} - z^{n-q_n}}{(v-u)w^{n-q_n+1}} \right) \sum_{j=1}^n \frac{c + \frac{v}{n} - x_j}{(z + c + \frac{v}{n} - x_j)^2} \prod_{i \neq j} \left(\frac{w + c + \frac{v}{n} - x_i}{z + c + \frac{v}{n} - x_i} \right), \end{aligned}$$

for all n sufficiently large, $u \in U$ and $v \in V$. Recall that, for all n sufficiently large and $v \in V$, $\Gamma_{n,2}^+(s) = \Gamma(w_{n,v} - \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}, s)$ for all $s \in [t_0, 1)$ and $\Gamma_{n,4}^+(s) = \Gamma(w_{n,v} + \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}, s)$ for all $s \in (1, T_0)$, where Γ is given in equation (3.12). Equations (3.12) and (3.19) imply that there exists an $\epsilon > 0$ for which

$$\inf_{v \in V} \inf_{x \in \mathbb{R}, z \in (\Gamma_n^{2,4})^*} |z - x| \geq t_0 \epsilon,$$

for all t_0 sufficiently small and n sufficiently large (chosen independently). Therefore we can choose the constant $C > 0$, and the function $N : (0, \infty) \rightarrow \mathbb{N}$, so that

$$\begin{aligned} & \left| K_{n,u,v}^{\{1,2,3,4,5\},\{2,4\}} \right| \\ & \leq C \frac{T_0^2}{t_0} \left(1 + e^{Ct_0^{-1}} \right) \sup_{(w,z) \in \gamma_n^* \times (\Gamma_n^{2,4})^*} \frac{|z|^{n-q_n}}{|w|^{n-q_n}} \sum_{j=1}^n \frac{1}{|z + c + \frac{v}{n} - x_j|} \prod_{i \neq j} \left| \frac{w + c + \frac{v}{n} - x_i}{z + c + \frac{v}{n} - x_i} \right|, \end{aligned}$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. Equations (3.3), (3.11) and (3.19) then show that we can choose C and N so that

$$(3.21) \quad \left| K_{n,u,v}^{\{1,2,3,4,5\},\{2,4\}} \right| \leq C n^2 \frac{T_0^2}{t_0} \left(1 + e^{Ct_0^{-1}} \right) \sup_{(w,z) \in (\gamma_n^{2,3,4})^* \times (\Gamma_n^{2,4})^*} \left| e^{n(h_{n,v}(w) - h_{n,v}(z))} \right|,$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. Similarly we can choose C so that

$$(3.22) \quad \left| K_{n,u,v}^{\{1,2,4,5\},3} \right| \leq C n^2 \sup_{(w,z) \in (\gamma_n^{2,4})^* \times (\Gamma_n^3)^*} \left| e^{n(h_{n,v}(w) - h_{n,v}(z))} \right|,$$

for all n sufficiently large, $u \in U$ and $v \in V$.

Consider $K_{n,u,v}^{\{1,2,3,4,5\},1}$. Recall that $\Gamma_{n,1}^+(s) = (1-s)y_{n,u,v} + s \Gamma(w_{n,v} - \varepsilon_0 n^{-\delta} e^{i\theta_{n,v}}, t_0)$ for all $s \in [0, 1)$, where $y_{n,u,v} \in (x_j - \frac{v}{n} - c, x_{j-1} - \frac{v}{n} - c)$ for that value of j which satisfies $c + \frac{u}{n} \in [x_j, x_{j-1})$ (see equation (3.13)). Also recall that $|G(w, s)| = s|w|$ for all $s > 0$ and $w \in \mathbb{C}$ with $\text{Im}(w) > 0$. It thus follows from equation (3.19) that we can choose $y_{n,u,v}$ so that

$$(3.23) \quad \sup_{u \in U, v \in V} \sup_{z \in (\Gamma_n^1)^*} |z| \leq 2t_0|w_0|,$$

for all n sufficiently large. Thus there exists a choice of $N : (0, \infty) \rightarrow \mathbb{N}$ for which the following is well-defined for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$:

$$\int_{\gamma_n} dw \int_{\Gamma_n^1} dz \frac{(z + \frac{v-u}{n})^{n-q_n-1}}{w^{n-q_n+1}} \frac{1}{w-z} \prod_{i=1}^n \left(\frac{w+c-\frac{v}{n}-x_i}{z+c-\frac{v}{n}-x_i} \right).$$

Using Cauchy's Theorem to perturb the contours in a similar manner to that described in Proposition 2.4, this quantity can be related to $K_{n,u,v}^{\{1,2,3,4,5\},1}$ in the following way:

$$\begin{aligned} K_{n,u,v}^{\{1,2,3,4,5\},1} &= \left(\frac{n-q_n}{n} \right) \int_{\gamma_n} dw \int_{\Gamma_n^1} dz \frac{(z + \frac{v-u}{n})^{n-q_n-1}}{w^{n-q_n+1}} \frac{1}{w-z} \prod_{i=1}^n \left(\frac{w+c-\frac{v}{n}-x_i}{z+c-\frac{v}{n}-x_i} \right) \\ &\quad - \int_{\gamma_n} dw \left(\frac{(\Gamma_{n,1}^-(0) + \frac{v-u}{n})^{n-q_n} - \Gamma_{n,1}^-(0)^{n-q_n}}{(v-u)w^{n-q_n+1}} \right) \frac{\Gamma_{n,1}^-(0)}{w - \Gamma_{n,1}^-(0)} \prod_{i=1}^n \left(\frac{w+c-\frac{v}{n}-x_i}{\Gamma_{n,1}^-(0) + c - \frac{v}{n} - x_i} \right) \\ &\quad + \int_{\gamma_n} dw \left(\frac{(\Gamma_{n,1}^+(1) + \frac{v-u}{n})^{n-q_n} - \Gamma_{n,1}^+(1)^{n-q_n}}{(v-u)w^{n-q_n+1}} \right) \frac{\Gamma_{n,1}^+(1)}{w - \Gamma_{n,1}^+(1)} \prod_{i=1}^n \left(\frac{w+c-\frac{v}{n}-x_i}{\Gamma_{n,1}^+(1) + c - \frac{v}{n} - x_i} \right), \end{aligned}$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. Therefore we can choose C and N so that

$$\left| K_{n,u,v}^{\{1,2,3,4,5\},1} \right| \leq C^n \sup_{z \in (\Gamma_n^1)^*} \left| z + \frac{v-u}{n} \right|^{n-q_n-1} \prod_{i=1}^n \frac{1}{|z+c-\frac{v}{n}-x_i|},$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. It thus follows from equation (3.23) that, for any fixed $\epsilon > 0$, we can choose C and N so that

$$\left| K_{n,u,v}^{\{1,2,3,4,5\},1} \right| \leq C^n \sup_{z \in (\Gamma_n^1)^*} \left| z + \frac{v-u}{n} \right|^{n-q_n-1} \prod_{i: |x_i-c| < \epsilon} \frac{1}{|z+c-\frac{v}{n}-x_i|},$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. Recalling that $c \in A_\alpha$, and so $\mu[\{c\}] < 1 - \alpha$ (see the proof of Proposition 1.7), we fix $\epsilon > 0$ so that $\frac{1}{n} \# \{i : |x_i - c| < \epsilon\} < \frac{1}{2}(1 - \alpha + \mu[\{c\}])$ for all n sufficiently large (which is always possible by hypothesis 1.1). Write

$$\left| K_{n,u,v}^{\{1,2,3,4,5\},1} \right| \leq C^n \sup_{z \in (\Gamma_n^1)^*} \left| z + \frac{v-u}{n} \right|^{n-q_n-1-\#\{i: |x_i-c| < \epsilon\}} \prod_{i: |x_i-c| < \epsilon} \left| \frac{z + \frac{v-u}{n}}{z+c-\frac{v}{n}-x_i} \right|,$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. Equation (3.23) then shows that we can choose C and N so that

$$\left| K_{n,u,v}^{\{1,2,3,4,5\},1} \right| \leq C^n (t_0)^{\frac{n}{2}(1-\alpha-\mu[\{c\}])} \sup_{z \in (\Gamma_n^1)^*} \prod_{i: |x_i-c| < \epsilon} \left(1 + \left| \frac{\frac{v-u}{n} - (c - \frac{v}{n} - x_i)}{z+c-\frac{v}{n}-x_i} \right| \right),$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. Moreover, for any fixed $\xi > 0$, equations (3.12) and (3.13) imply that we can choose N so that $\sup_{v \in V} |\text{Arg}(\Gamma_{n,1}^+(1)) - \frac{\pi}{2}| \leq \xi$ for all t_0 sufficiently small and $n \geq N(t_0)$. It thus follows that we can choose N so that

$$\begin{aligned} \left| K_{n,u,v}^{\{1,2,3,4,5\},1} \right| &\leq C^n (t_0)^{\frac{n}{2}(1-\alpha-\mu[\{c\}])} \prod_{i: |x_i-c| < \epsilon} \left(1 + 2 \left| \frac{\frac{v-u}{n} - (c - \frac{v}{n} - x_i)}{y_{n,u,v} + c - \frac{v}{n} - x_i} \right| \right) \\ &\leq C^n (t_0)^{\frac{n}{2}(1-\alpha-\mu[\{c\}])} \prod_{i: |x_i-c| < \epsilon} \left(1 + 2 \left(1 + \left| \frac{y_{n,u,v} + \frac{v-u}{n}}{y_{n,u,v} + c - \frac{v}{n} - x_i} \right| \right) \right), \end{aligned}$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. Finally, choosing $y_{n,u,v}$ sufficiently close to $\frac{u-v}{n}$ (recall that $y_{n,u,v} \in (x_j - \frac{v}{n} - c, x_{j-1} - \frac{v}{n} - c)$ for that value of j which satisfies $c + \frac{u}{n} \in [x_j, x_{j-1})$),

$$(3.24) \quad \left| K_{n,u,v}^{\{1,2,3,4,5\},1} \right| \leq (5C)^n (t_0)^{\frac{n}{2}(1-\alpha-\mu[\{c\}]},$$

for all t_0 sufficiently small, $n \geq N(t_0)$, $u \in U$ and $v \in V$. The required result follows from equations (3.20), (3.21), (3.22) and (3.24) by fixing t_0 sufficiently small and T_0 sufficiently large. \square

Lemma 3.3. *For n all sufficiently large and $v \in V$,*

$$\sup_{w \in (\gamma_n^3)^*} \operatorname{Re}(h_{n,v}(w) - h_{n,v}(w_{n,v})) = 0.$$

Moreover, letting $\delta \in (\frac{1}{3}, \frac{1}{2})$ be that used in equations (3.11) and (3.13), there exists a constant $c > 0$ for which

$$\lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{w \in (\gamma_n^{2,4})^*} n^{2\delta} \operatorname{Re}(h_{n,v}(w) - h_{n,v}(w_{n,v})) \leq -c.$$

PROOF. Consider γ_n^3 . For all n sufficiently large and $v \in V$ define $f_{n,v} : [-1, 1] \rightarrow \mathbb{R}$ by $f_{n,v}(s) := \operatorname{Re}(h_{n,v}(\gamma_{n,3}^+(s)))$ for all $s \in [-1, 1]$. Equation (3.11) gives $f'_{n,v}(0) = 0$ for all n sufficiently large and $v \in V$. Also Lemma 3.1 and equation (3.7) give $h''(w_0) \neq 0$ and

$$\sup_{s \in [-1, 1]} \sup_{v \in V} f''_{n,v}(s) \leq -n^{-2\delta} \sup_{v \in V} \sup_{w \in B(w_{n,v}, n^{-\delta})} \operatorname{Re} \left(h''_{n,v}(w) e^{-i \operatorname{Arg}(h''_{n,v}(w_{n,v}))} \right) \leq -\frac{1}{2} n^{-2\delta} |h''(w_0)|,$$

for all n sufficiently large, as required.

Now consider $\gamma_n^{2,4}$. Define $z_{n,v}^\pm := w_{n,v} \pm i n^{-\delta} e^{i\theta_{n,v}}$ for all n sufficiently large and $v \in V$, where $\theta_{n,v} = -\frac{1}{2} \operatorname{Arg}(h''_{n,v}(w_{n,v}))$. Also define $f, f_{n,v}^\pm : (0, \pi) \rightarrow \mathbb{R}$ by

$$(3.25) \quad f(s) := \operatorname{Re}(|w_0| e^{is}) \quad \text{and} \quad f_{n,v}^\pm(s) := \operatorname{Re}(h_{n,v}(|z_{n,v}^\pm| e^{is})),$$

for all n sufficiently large, $v \in V$ and $s \in (0, \pi)$. Equations (3.3) and (3.6) give

$$\begin{aligned} f(s) &= \frac{1}{2} \int \log ||w_0| e^{is} + c - x|^2 \mu[dx] - (1 - \alpha) \log |w_0|, \\ f_{n,v}^\pm(s) &= \frac{1}{2} \int \log ||z_{n,v}^\pm| e^{is} + c - x|^2 \mu_{n,v}[dx] - \frac{n - q_n}{n} \log |z_{n,v}^\pm|, \end{aligned}$$

for all n sufficiently large, $v \in V$ and $s \in (0, \pi)$. It is easy to see that $f''(s) < 0$ for any $s \in (0, \pi)$ with $f'(s) = 0$. Also equation (3.8) gives $f'(\operatorname{Arg}(w_0)) = 0$. Thus f has a unique critical point in $(0, \pi)$, a global maximum at $\operatorname{Arg}(w_0)$. Similarly, for all n sufficiently large and $v \in V$, $f_{n,v}^\pm$ has at most one critical point in $(0, \pi)$ which, if it exists, must be a global maximum. To demonstrate its existence, fix $\phi_0 \in (0, \frac{\pi}{2})$. Since $\frac{q_n}{n} \rightarrow \alpha$ and $\mu_{n,0} \rightarrow \mu$ weakly as $n \rightarrow \infty$ (see equation (3.5) and hypothesis 1.1), Lemma 3.1 implies that

$$\lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{s \in [\phi_0, \pi - \phi_0]} |f_{n,v}^\pm(s) - f(s)| = 0.$$

Thus, for all n sufficiently large and $v \in V$, since the unique critical point of f in $(0, \pi)$ is a global maximum at $\operatorname{Arg}(w_0)$, $f_{n,v}^\pm$ must have a unique critical point in $(0, \pi)$. Denoting by $s_{n,v}^\pm$, it also follows that $\sup_{v \in V} |s_{n,v}^\pm - \operatorname{Arg}(w_0)| \rightarrow 0$ as $n \rightarrow \infty$.

Equation (3.8) gives

$$(f_{n,v}^\pm)'(\operatorname{Arg}(z_{n,v}^\pm)) = \operatorname{Im}(z_{n,v}^\pm) \int \left(\frac{c - x}{|w_{n,v} + c - x|^2} - \frac{c - x}{|z_{n,v}^\pm + c - x|^2} \right) \mu_{n,v}[dx],$$

for all n sufficiently large and $v \in V$. Then, since $\mu_{n,0} \rightarrow \mu$ weakly as $n \rightarrow \infty$ (see equation (3.5) and hypothesis 1.1),

$$(3.26) \quad \lim_{n \rightarrow \infty} \sup_{v \in V} |n^\delta (f_{n,v}^\pm)'(\text{Arg}(z_{n,v}^\pm)) \mp C_0| = 0,$$

where

$$(3.27) \quad C_0 := 2\text{Im}(w_0) \int \frac{(c-x)\text{Im}((w_0+c-x)e^{\frac{i}{2}\text{Arg}(h''(w_0))})}{|w_0+c-x|^4} \mu[dx].$$

Also equation (3.8) gives

$$(f_{n,v}^\pm)''(s) = \int \left(\frac{(c-x)|z_{n,v}^\pm| \cos(s)}{|w_{n,v}+c-x|^2} - \frac{(c-x)|z_{n,v}^\pm| \cos(s)}{||z_{n,v}^\pm|e^{is}+c-x|^2} - 2 \left(\frac{(c-x)|z_{n,v}^\pm| \sin(s)}{||z_{n,v}^\pm|e^{is}+c-x|^2} \right)^2 \right) \mu_{n,v}[dx],$$

for all n sufficiently large, $v \in V$ and $s \in (0, \pi)$. Thus there exists a constant $C > 0$ for which

$$\left| (f_{n,v}^\pm)''(s) + 2\text{Im}(w_0)^2 \int \frac{(c-x)^2}{|w_0+c-x|^4} \mu_{n,v}[dx] \right| \leq C(|s - \text{Arg}(w_0)| + |w_{n,v} - w_0| + n^{-\delta}),$$

for all n sufficiently large, $v \in V$ and s sufficiently close to $\text{Arg}(w_0)$. Thus, since $\mu_{n,0} \rightarrow \mu$ weakly as $n \rightarrow \infty$ (see equation (3.5) and hypothesis 1.1), Lemma 3.1 implies that there exists an $\epsilon > 0$ for which $(f_{n,v}^\pm)''(s) < -\epsilon$ for all n sufficiently large, $v \in V$ and s sufficiently close to $\text{Arg}(w_0)$.

Consider the case $C_0 > 0$. Then $\varepsilon_0 = -1$ (recall $\varepsilon_0 = (-1)^{1_{C_0 \geq 0}}$) and, for all n sufficiently large and $v \in V$, equations (3.11) and (3.25) give

$$\begin{aligned} \sup_{w \in (\gamma_n^2)^*} \text{Re}(h_{n,v}(w) - h_{n,v}(\gamma_{n,3}^+(-1))) &= \sup_{s \in [\frac{1}{n}, \text{Arg}(z_{n,v}^+)]} (f_{n,v}^+(s) - f_{n,v}^+(\text{Arg}(z_{n,v}^+))), \\ \sup_{w \in (\gamma_n^4)^*} \text{Re}(h_{n,v}(w) - h_{n,v}(\gamma_{n,3}^+(1))) &= \sup_{s \in [\text{Arg}(z_{n,v}^-), \pi - \frac{1}{n}]} (f_{n,v}^-(s) - f_{n,v}^-(\text{Arg}(z_{n,v}^-))). \end{aligned}$$

Also equation (3.26) gives $(f_{n,v}^-)'(\text{Arg}(z_{n,v}^-)) < 0 < (f_{n,v}^+)'\text{Arg}(z_{n,v}^+)$ for all n sufficiently large and $v \in V$. Recall that the unique critical point of $f_{n,v}^\pm$ in $(0, \pi)$ is a global maximum at $s_{n,v}^\pm$. Thus for all n sufficiently large and $v \in V$, $\text{Arg}(z_{n,v}^+) < s_{n,v}^+$, $\text{Arg}(z_{n,v}^-) > s_{n,v}^-$, and

$$\sup_{w \in (\gamma_n^2)^*} \text{Re}(h_{n,v}(w) - h_{n,v}(\gamma_{n,3}^+(-1))) = 0 = \sup_{w \in (\gamma_n^4)^*} \text{Re}(h_{n,v}(w) - h_{n,v}(\gamma_{n,3}^+(1))).$$

It thus follows that

$$\sup_{w \in (\gamma_n^{2,4})^*} \text{Re}(h_{n,v}(w) - h_{n,v}(w_{n,v})) = \text{Re}(h_{n,v}(\gamma_{n,3}^+(\pm 1)) - h_{n,v}(w_{n,v})),$$

for all n sufficiently large and $v \in V$. Thus, since $\delta \in (\frac{1}{3}, \frac{1}{2})$ and $h''(w_0) \neq 0$ (see Lemma 3.1), equations (3.10), (3.17) and (3.18) give the required result. Similarly for $C_0 < 0$.

Now suppose $C_0 = 0$. Recall that there exists an $\epsilon > 0$ for which $(f_{n,v}^\pm)''(s) < -\epsilon$ for all n sufficiently large, $v \in V$ and s sufficiently close to $\text{Arg}(w_0)$. Thus, since $\sup_{v \in V} |s_{n,v}^\pm - \text{Arg}(w_0)| \rightarrow 0$ and $\sup_{v \in V} |z_{n,v}^\pm - w_0| \rightarrow 0$ as $n \rightarrow \infty$,

$$\epsilon |s_{n,v}^\pm - \text{Arg}(z_{n,v}^\pm)| \leq |(f_{n,v}^\pm)'(\text{Arg}(z_{n,v}^\pm))|,$$

for all n sufficiently large and $v \in V$. Moreover

$$f_{n,v}^\pm(s_{n,v}^\pm) - f_{n,v}^\pm(\text{Arg}(z_{n,v}^\pm)) \leq |s_{n,v}^\pm - \text{Arg}(z_{n,v}^\pm)| |(f_{n,v}^\pm)'(\text{Arg}(z_{n,v}^\pm))|,$$

for all n sufficiently large and $v \in V$, and so

$$f_{n,v}^\pm(s_{n,v}^\pm) - f_{n,v}^\pm(\text{Arg}(z_{n,v}^\pm)) \leq \epsilon^{-1} |(f_{n,v}^\pm)'(\text{Arg}(z_{n,v}^\pm))|^2.$$

Thus, since $C_0 = 0$, equation (3.26) gives

$$\lim_{n \rightarrow \infty} \sup_{v \in V} n^{2\delta} (f_{n,v}^\pm(s_{n,v}^\pm) - f_{n,v}^\pm(\text{Arg}(z_{n,v}^\pm))) = 0.$$

Then, recalling that $s_{n,v}^\pm$ is the global maximum of $f_{n,v}^\pm$, equation (3.25) gives

$$\lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{s \in (0, \pi)} n^{2\delta} \text{Re}(h_{n,v}(|z_{n,v}^\pm|e^{is}) - h_{n,v}(z_{n,v}^\pm)) = 0.$$

The required result follows from equations (3.11) and (3.17) in a similar way to before. \square

Lemma 3.4. *For all n sufficiently large and $v \in V$,*

$$\sup_{z \in (\Gamma_n^3)^*} \text{Re}(h_{n,v}(w_{n,v}) - h_{n,v}(z)) = 0.$$

Moreover, letting $\delta \in (\frac{1}{3}, \frac{1}{2})$ be that used in equations (3.11) and (3.13), there exists a constant $c > 0$ for which

$$\lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{z \in (\Gamma_n^{2,4})^*} n^{2\delta} \text{Re}(h_{n,v}(w_{n,v}) - h_{n,v}(z)) \leq -c.$$

PROOF. The result for Γ_n^3 follows in a similar way to that for γ_n^3 given in Lemma 3.3. Consider $\Gamma_n^{2,4}$. For all n sufficiently large and $v \in V$, define $z_{n,v}^\pm := w_{n,v} \pm n^{-\delta} e^{i\theta_{n,v}^\pm}$, and $p, p_{n,v}^\pm : (0, \infty) \rightarrow \mathbb{R}$ by

$$p(s) := -\text{Re}(h(\Gamma(w_0, s))) \quad \text{and} \quad p_{n,v}^\pm(s) := -\text{Re}(h_{n,v}(\Gamma_{n,v}^\pm(z_{n,v}^\pm, s))),$$

for all $s > 0$, where Γ is defined in equation (3.12). Equations (3.3) and (3.6) give

$$\begin{aligned} p(s) &= -\frac{1}{2} \int \log |\Gamma(w_0, s) + c - x|^2 \mu[dx] + (1 - \alpha) \log |\Gamma(w_0, s)|, \\ p_{n,v}^\pm(s) &= -\frac{1}{2} \int \log |\Gamma(z_{n,v}^\pm, s) + c - x|^2 \mu_{n,v}[dx] + \frac{n - q_n}{n} \log |\Gamma(z_{n,v}^\pm, s)|, \end{aligned}$$

for all n sufficiently large, $v \in V$ and $s > 0$.

Consider p . Recalling that $|\Gamma(w_0, s)| = s|w_0|$ for all $s > 0$ (see equation (3.12)), equations (3.8) and (3.9) give

$$p'(s) = \int \left(-\frac{s|w_0|^2 + (c - x) \frac{d}{ds} \text{Re}(\Gamma(w_0, s))}{|\Gamma(w_0, s) + c - x|^2} + \frac{|w_0|^2 + (c - x)g(s)}{s|w_0 + c - x|^2} \right) \mu[dx],$$

for all $s > 0$, where $g : (0, \infty) \rightarrow \mathbb{R}$ is arbitrary. Taking $g(s) := s \frac{d}{ds} \text{Re}(\Gamma(w_0, s))$ for all $s > 0$, equation (3.12) gives

$$p'(s) = \int \frac{q(s)(c - x)^2}{s|\Gamma(w_0, s) + c - x|^2 |w_0 + c - x|^2} \mu[dx],$$

for all $s > 0$, where

$$q(s) := (1 - s^2) \text{Im}(w_0)^2 + \frac{1}{1 - s^2} \left(1 + s^2 + \frac{4s^2}{1 - s^2} \log(s) \right)^2 \text{Re}(w_0)^2.$$

Note that $q(s) > 0$ for all $s \in (0, 1)$, $q(1) = 0$, $q(s) < 0$ for all $s > 1$, and $q'(1) = -2\text{Im}(w_0)^2$. Therefore p is strictly increasing in $(0, 1)$, strictly decreasing in $(1, \infty)$, and has a global maximum at 1.

Consider $p_{n,v}^\pm$. Proceeding in a similar way to that given above,

$$(3.28) \quad (p_{n,v}^\pm)'(s) = \int \frac{r_{n,v}^\pm(s)(c - x) + q_{n,v}^\pm(s)(c - x)^2}{s|\Gamma(z_{n,v}^\pm, s) + c - x|^2 |w_{n,v} + c - x|^2} \mu_{n,v}[dx],$$

for all n sufficiently large, $v \in V$ and $s > 0$, where

$$\begin{aligned} r_{n,v}^\pm(s) &:= 2s^2 |z_{n,v}^\pm|^2 \operatorname{Re}(z_{n,v}^\pm - w_{n,v}) + \frac{2s^2}{s^2 - 1} \left(1 - \frac{2s^2 \log(s)}{s^2 - 1} \right) \operatorname{Re}(z_{n,v}^\pm) (|z_{n,v}^\pm|^2 - |w_{n,v}|), \\ q_{n,v}^\pm(s) &:= (1 - s^2) \operatorname{Im}(z_{n,v}^\pm)^2 + \frac{1}{1 - s^2} \left(1 + s^2 + \frac{4s^2}{1 - s^2} \log(s) \right)^2 \operatorname{Re}(z_{n,v}^\pm)^2 \\ &\quad + |w_{n,v}|^2 - |z_{n,v}^\pm|^2 + \frac{4s^2}{(s^2 - 1)^2} (2 \log(s) - s^2 + 1) \operatorname{Re}(z_{n,v}^\pm) \operatorname{Re}(w_{n,v} - z_{n,v}^\pm). \end{aligned}$$

These are well-defined and continuous with

$$(3.29) \quad \begin{aligned} r_{n,v}^\pm(1) &= 2|z_{n,v}^\pm|^2 \operatorname{Re}(z_{n,v}^\pm - w_{n,v}) - \operatorname{Re}(z_{n,v}^\pm) (|z_{n,v}^\pm|^2 - |w_{n,v}|^2), \\ q_{n,v}^\pm(1) &= |w_{n,v}|^2 - |z_{n,v}^\pm|^2 + 2\operatorname{Re}(z_{n,v}^\pm) \operatorname{Re}(z_{n,v}^\pm - w_{n,v}), \end{aligned}$$

for all n sufficiently large and $v \in V$. Also there exists a constant $C > 0$ for which

$$\left| (p_{n,v}^\pm)''(s) + 2\operatorname{Im}(w_0)^2 \int \frac{(c-x)^2}{|w_0 + c - x|^4} \mu_{n,v}[dx] \right| \leq C (|s-1| + |w_{n,v} - w_0| + n^{-\delta}),$$

for all n sufficiently large, $v \in V$ and s sufficiently close to 1. Thus, since $\mu_{n,0} \rightarrow \mu$ weakly as $n \rightarrow \infty$ (see equation (3.5) and hypothesis 1.1), Lemma 3.1 implies that there exists an $\epsilon > 0$ for which $(p_{n,v}^\pm)''(s) < -\epsilon$ for all n sufficiently large, $v \in V$ and s sufficiently close to 1. Also note, since $\frac{q_n}{n} \rightarrow \alpha$ and $\mu_{n,0} \rightarrow \mu$ weakly as $n \rightarrow \infty$, Lemma 3.1 gives

$$\lim_{n \rightarrow \infty} \sup_{v \in V} \sup_{s \in [t_0, T_0]} |p_{n,v}^\pm(s) - p(s)| = 0.$$

Thus for all n sufficiently large and $v \in V$, since the unique critical point of p in $(0, \infty)$ is a global maximum at 1, and since $t_0 < 1 < T_0$, p_n^\pm has a unique critical point in $[t_0, T_0]$ and this point is a local maximum. Denoting by $s_{n,v}^\pm$, it also follows that $\sup_{v \in V} |s_{n,v}^\pm - 1| \rightarrow 0$ as $n \rightarrow \infty$.

Since $\mu_{n,0} \rightarrow \mu$ weakly as $n \rightarrow \infty$, equations (3.28) and (3.29) give

$$\lim_{n \rightarrow \infty} \sup_{v \in V} |n^\delta (p_{n,v}^\pm)'(1) \mp C_0| = 0,$$

where $C_0 \in \mathbb{R}$ is defined in equation (3.27). The required result then follows from a similar argument to that given in Lemma 3.3. \square

4. Unitary invariant ensemble

In this section we consider measures on $\overline{\text{GT}}_n$ induced by the eigenvalue minor process of a Unitary invariant ensemble (UIE). As in the introduction, for each $n \in \mathbb{N}$, let $\mathcal{H}_n \subset \mathbb{C}^{n \times n}$ be the set of $n \times n$ complex Hermitian matrices. Let $A_n \in \mathcal{H}_n$ be a random Unitary matrix with eigenvalue distribution

$$d\mu_n[y] = \frac{1}{Z_n} \Delta_n(y)^2 \left(\prod_{i=1}^n e^{-V(y_i)} \right) dy,$$

for all $y \in \overline{\mathcal{C}}_n$, where $Z_n > 0$ is a normalisation constant, $V : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and dy is Lebesgue measure on \mathbb{R}^n . Then A_n is called a UIE with *potential* V . For more information on UIEs see Anderson, Guionnet and Zeitouni, [1], and Mehta, [20]. It follows from equation (1.6) that the GUE is the UIE with potential $V(x) = \frac{1}{2}x^2$ for all $x \in \mathbb{R}$.

Equations (1.3) and (1.7) imply that the eigenvalue minor process of A_n has distribution

$$d\nu_n[y^{(1)}, \dots, y^{(n)}] = \frac{1}{Z'_n} \Delta_n(y^{(n)}) \left(\prod_{i=1}^n e^{-V(y_i^{(n)})} \right) dy^{(n)} dy^{(n-1)} \dots dy^{(1)},$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\text{GT}}_n$, where $Z'_n > 0$ is a normalisation constant, and $dy^{(r)}$ is Lebesgue measure on \mathbb{R}^r for each r . We now use Theorem 2.1 to show that $(\overline{\text{GT}}_n, \nu_n)$ is a determinantal random point field, and obtain an expression for the correlation kernel in terms of polynomials which are orthogonal with respect to the weight $e^{-V(\cdot)} : \mathbb{R} \rightarrow \mathbb{R}_+$. We specialise to classical ensembles in Section 4.1.

Proposition 4.1. *For each $i, j \geq 0$, let ψ_i, ψ_j be the monic polynomials of degree i and j (respectively) which satisfy*

$$(4.1) \quad \int_{-\infty}^{\infty} dx \psi_i(x) \psi_j(x) e^{-V(x)} = c_i c_j \delta_{ij},$$

for some $c_i, c_j \in \mathbb{R}$. Then the random point field $(\overline{\text{GT}}_n, \nu_n)$ is determinantal with correlation kernel $K_n : (\{1, \dots, n\} \times \mathbb{R})^2 \rightarrow \mathbb{C}$ given by

$$\begin{aligned} K_n((r, u), (s, v)) &= 1_{v \leq u} \sum_{j=n-s}^{n-1} c_j^{-2} \psi_j^{(n-s)}(v) \int_u^{\infty} dx \frac{(x-u)^{n-r-1}}{(n-r-1)!} \psi_j(x) e^{-V(x)} \\ &\quad - 1_{v > u} \sum_{j=n-s}^{n-1} c_j^{-2} \psi_j^{(n-s)}(v) \int_{-\infty}^u dx \frac{(x-u)^{n-r-1}}{(n-r-1)!} \psi_j(x) e^{-V(x)}, \end{aligned}$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$, where $\psi_j^{(i)}$ is the i^{th} derivative of ψ_j for each i, j .

PROOF. It follows from equation (1.7) that

$$d\nu_n[y^{(1)}, \dots, y^{(n)}] = \frac{1}{Z'_n} \det \left[\psi_{n-l}(y_m^{(n)}) e^{-V(y_m^{(n)})} \right]_{l,m=1}^n dy^{(n)} \dots dy^{(1)},$$

for all $(y^{(1)}, \dots, y^{(n)}) \in \overline{\text{GT}}_n$. This is written in the form of equation (2.1) with $\phi_i(x) = \psi_{n-i}(x) e^{-V(x)}$ for all $i \in \{1, \dots, n\}$ and $x \in \mathbb{R}$. The fact that $(\overline{\text{GT}}_n, \nu_n)$ is determinantal follows immediately from Theorem 2.1. The correlation kernel $K_n : (\{1, \dots, n\} \times \mathbb{R})^2 \rightarrow \mathbb{C}$ is given by

$$\begin{aligned} K_n((r, u), (s, v)) &= \frac{1}{B_n} \frac{\partial^{n-s}}{\partial v^{n-s}} \sum_{j=1}^n \int_{\mathbb{R}^n} dy \left(\prod_{k=1}^n \psi_{n-k}(y_k) e^{-V(y_k)} \right) 1_{v \leq u < y_j} \frac{(y_j - u)^{n-r-1}}{(n-r-1)!} \Delta_n(y_{j,v}) \\ &\quad - \frac{1}{B_n} \frac{\partial^{n-s}}{\partial v^{n-s}} \sum_{j=1}^n \int_{\mathbb{R}^n} dy \left(\prod_{k=1}^n \psi_{n-k}(y_k) e^{-V(y_k)} \right) 1_{v > u \geq y_j} \frac{(y_j - u)^{n-r-1}}{(n-r-1)!} \Delta_n(y_{j,v}), \end{aligned}$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$, where $y_{j,v} := (y_1, \dots, y_{j-1}, v, y_{j+1}, \dots, y_n)$ and

$$B_n = \int_{\mathbb{R}^n} dy \left(\prod_{k=1}^n \psi_{n-k}(y_k) e^{-V(y_k)} \right) \Delta_n(y).$$

Then, writing $\Delta_n(y) = \det[\psi_{n-l}(y_m)]_{l,m=1}^n$ (see equation (1.7)), equation (4.1) gives $B_n = \prod_{k=0}^{n-1} c_k^2$. Similarly, writing $\Delta_n(y_{j,v}) = \det[\psi_{n-l}(y_m) 1_{m \neq j} + \psi_{n-l}(v) 1_{m=j}]_{l,m=1}^n$, equation (4.1) gives the required result. \square

Alternatively the correlation kernel in the previous Proposition can be written as a contour integral. Using the choice of $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$ given in the previous Proposition, it follows from Proposition 2.4 that

$$\begin{aligned} K_n((r, u), (s, v)) &= \frac{1}{(2\pi)^2} \frac{(n-s)!}{(n-r-1)!} \int_{\gamma_{u,v}} dw \int_{\Gamma_{u,v}} dz \frac{(z-u)^{n-r-1}}{(w-v)^{n-s+1}} \frac{1}{w-z} \times \\ &\times \frac{1}{B_n} \int_{\mathbb{R}^n} dy \left(\prod_{k=1}^n \psi_{n-k}(y_k) e^{-V(y_k)} \right) \Delta_n(y) \prod_{i=1}^n \left(\frac{w-y_i}{z-y_i} \right), \end{aligned}$$

for all $r \in \{1, \dots, n-2\}$, $s \in \{1, \dots, n\}$, and $u, v \in \mathbb{R}$ with $u \neq v$. Here $\gamma_{u,v}$ is a counter-clockwise simple closed contour around v . Also $\Gamma_{u,v} : \mathbb{R} \rightarrow \mathbb{C}$ is a piecewise smooth contour which satisfies $\Gamma_{u,v}(0) = u$, $\text{Im}(\Gamma_{u,v}(s)) > 0$ for all $s > 0$, $\text{Im}(\Gamma_{u,v}(s)) < 0$ for all $s < 0$, and $\lim_{s \rightarrow \pm\infty} |\Gamma_{u,v}(s)| = \infty$. Moreover the contours are chosen not to intersect. Then, recalling that $\Delta_n(y) = \det[\psi_{n-l}(y_m)]_{l,m=1}^n$ for all $y \in \mathbb{R}^n$,

$$\begin{aligned} K_n((r, u), (s, v)) &= \frac{1}{(2\pi)^2} \frac{(n-s)!}{(n-r-1)!} \int_{\gamma_{u,v}} dw \int_{\Gamma_{u,v}} dz \frac{(z-u)^{n-r-1}}{(w-v)^{n-s+1}} \frac{1}{w-z} \times \\ &\times \frac{1}{n! B_n} \int_{\mathbb{R}^n} dy \left(\prod_{k=1}^n e^{-V(y_k)} \right) \Delta_n(y)^2 \prod_{i=1}^n \left(\frac{w-y_i}{z-y_i} \right), \end{aligned}$$

for all $r \in \{1, \dots, n-2\}$, $s \in \{1, \dots, n\}$, and $u, v \in \mathbb{R}$ with $u \neq v$. Then, recalling that $B_n = \prod_{k=0}^{n-1} c_k^2$ (see proof of Proposition 4.1), Fyodorov and Strahov, [13], gives

$$\begin{aligned} K_n((r, u), (s, v)) &= \frac{1}{(2\pi)^2} \frac{(n-s)!}{(n-r-1)!} \int_{\gamma_{u,v}} dw \int_{\Gamma_{u,v}} dz \frac{(z-u)^{n-r-1}}{(w-v)^{n-s+1}} \frac{1}{w-z} \times \\ &\times c_{n-1}^{-2} \int_{-\infty}^{\infty} dx e^{-V(x)} \frac{\psi_{n-1}(x) \psi_n(w) - \psi_n(x) \psi_{n-1}(w)}{z-x}, \end{aligned}$$

for all $r \in \{1, \dots, n-2\}$, $s \in \{1, \dots, n\}$, and $u, v \in \mathbb{R}$ with $u \neq v$.

4.1. The classical ensembles. We end this paper by showing that the expression for the correlation kernel obtained in Proposition 4.1 in the special case of classical UIEs, agrees with the expression obtained by Johansson and Nordenstam, [17], for the GUE (see equation (1.9)). By classical UIEs we mean those that satisfy the *generalised Rodrigues formula*:

Hypothesis 4.1. *There exists a function $\alpha : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ for which $\lim_{x \rightarrow \pm\infty} x^j \alpha(x)^k e^{-V(x)} = 0$ for all $j, k \in \mathbb{N}$. Also there exists and a sequence, $\{a_{j,k}\}_{j,k \in \mathbb{N}} \subset \mathbb{R} \setminus \{0\}$, for which*

$$\psi_j^{(k)}(x) = a_{j,k} \alpha(x)^{-k} e^{V(x)} \frac{d^{j-k}}{dx^{j-k}} \left(\alpha(x)^j e^{-V(x)} \right),$$

for all $x \in \mathbb{R}$ and $j, k \in \mathbb{N}$ with $j \geq k$.

First note an alternative expression for the correlation kernel in Proposition 4.1 can be obtained using equation (3.1.12) of Szegő, [28]. This gives

$$1_{s>r} \frac{(v-u)^{s-r-1}}{(s-r-1)!} = \frac{\partial^{n-s}}{\partial v^{n-s}} \int_{-\infty}^{\infty} dx \frac{(x-u)^{n-r-1}}{(n-r-1)!} \left(\sum_{j=0}^{n-1} c_j^{-2} \psi_j(x) \psi_j(v) e^{-V(x)} \right),$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$. Proposition 4.1 then implies that

$$K_n((r, u), (s, v)) = \sum_{j=n-s}^{n-1} c_j^{-2} \psi_j^{(n-s)}(v) \int_u^{\infty} dx \frac{(x-u)^{n-r-1}}{(n-r-1)!} \psi_j(x) e^{-V(x)} - \frac{(v-u)^{s-r-1}}{(s-r-1)!} 1_{v>u} 1_{s>r},$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$. Applying the Rodrigues formula inside the integral, and integrating by parts gives

$$K_n((r, u), (s, v)) = (-1)^{n-r} \sum_{j=-\infty}^{n-1} \left(c_j^{-2} \frac{a_{j,0}}{a_{j,n-r}} \right) \psi_j^{(n-r)}(u) \psi_j^{(n-s)}(v) \alpha(u)^{n-r} e^{-V(u)} \\ + \mathbf{1}_{s>r} \left(\sum_{j=n-s}^{n-r-1} (-1)^j c_j^{-2} a_{j,0} \psi_j^{(n-s)}(v) \int_u^\infty dx \frac{(x-u)^{n-r-1-j}}{(n-r-1-j)!} \alpha(x)^j e^{-V(x)} - \frac{(v-u)^{s-r-1}}{(s-r-1)!} \mathbf{1}_{v>u} \right),$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$.

In the special cases of the GUE, Laguerre and Jacobi ensembles, the above correlation kernel agrees with that given in equation (4.15) of Forrester and Nagao, [12]. In particular, in the GUE case, recall that (see, for example, Anderson, Guionnet and Zeitouni, [1]) for all $j \in \mathbb{N}$ and $x \in \mathbb{R}$, $\psi_j = H_j$ (the monic Hermite polynomial of degree j), $c_j = \sqrt{\sqrt{2\pi} j!}$, $\alpha(x) = 1$, and $a_{j,k} = (-1)^{j-k} \frac{j!}{(j-k)!}$ for all $k \leq j$. Also $H'_{j+1}(x) = (j+1)H_j(x)$ for all $j \in \mathbb{N}$ and $x \in \mathbb{R}$. Therefore

$$K_n((r, u), (s, v)) = \sum_{j=-\infty}^{-1} \frac{1}{\sqrt{2\pi}(j+s)!} H_{j+r}(u) H_{j+s}(v) e^{-\frac{1}{2}u^2} \\ + \mathbf{1}_{s>r} \left(\sum_{j=-s}^{-r-1} \frac{1}{\sqrt{2\pi}(j+s)!} H_{j+s}(v) \int_u^\infty dx \frac{(x-u)^{-r-1-j}}{(-r-1-j)!} e^{-\frac{1}{2}x^2} - \frac{(v-u)^{s-r-1}}{(s-r-1)!} \mathbf{1}_{v>u} \right),$$

for all $r \in \{1, \dots, n-1\}$, $s \in \{1, \dots, n\}$ and $u, v \in \mathbb{R}$. This agrees with the kernel given in equation (1.9) (see remark 1.2), and with equation (4.15) of Forrester and Nagao, [12]. Similarly for Jacobi and Laguerre ensembles.

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References

- [1] G. Anderson and A. Guionnet, and O. Zeitouni. An Introduction to Random Matrices. *Cambridge University Press*. (2010).
- [2] Y. Baryshnikov. GUEs and queues. *Probab. Theory Related Fields*. 119 (2001) 256–274.
- [3] S. Belinschi. The atoms of the free multiplicative convolution of two probability distributions. *Integral Equations and Operator Theory*. 46,4 (2003) 377–386.
- [4] Ph.D. thesis of S. Belinschi. Complex analysis methods in noncommutative probability. *Indiana University*. (2006).
- [5] C. Boutillier. The bead model and limit behaviors of dimer models. *Ann. Probab.* 37,1 (2009) 107–142.
- [6] B. Collins. Intégrales matricielles et probabilités non-commutatives. *Thèse de doctorat de l'Université Paris 6*, (2003).
- [7] B. Collins. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral and free probability. *IMRN*, 17, (2003), 953–982.
- [8] B. Collins. Product of random projections, Jacobi ensembles and universality problems arising from free probability. *Probability Theory and Related Fields* 3 (2005) 315–344.
- [9] L. Erdős. Universality of Wigner random matrices: A survey of recent results. *ArXiv e-prints*. arXiv:1004.0861v2 (2010).
- [10] P. Ferrari and H. Spohn. Step fluctuations for a faceted crystal. *Journal of Statistical Physics*. 113 (2003) 1–46.
- [11] B. Fleming and P. Forrester and E. Nordenstam. A finitization of the bead process. *Probability Theory and Related Fields*. 1,36 (2010).
- [12] P. Forrester and T. Nagao. Determinantal correlations for classical projection processes. *J. Stat. Mech.* P08011 (2011).

- [13] Y. Fyodorov and E. Strahov. An exact formula for general spectral correlation function of random Hermitian matrices. *J. Phys. A: Math. Gen.* 36 (2003) 3203–3214.
- [14] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press. (1990).
- [15] K. Johansson. Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. *Communications in Mathematical Physics*. 215 (2001) 683–705.
- [16] K. Johansson. Discrete polynuclear growth and determinantal processes. *Communications in Mathematical Physics*. 242 (2003) 277–329.
- [17] K. Johansson and E. Nordenstam. Eigenvalues of GUE Minors. *Electronic Journal of Probability*. 11 (2006) 1342–1371.
- [18] K. Johansson. Random matrices and determinantal processes. *Math. Stat. Phys, Session LXXXIII: Lecture Notes of the Les Houches Summer School, Elsevier Science*. (2006) 1–56.
- [19] M. Defosseux. Orbit measures, random matrix theory and interlaced determinantal processes. *ArXiv e-prints*. arXiv:0810.1011v2 (2008).
- [20] M. Mehta. *Random Matrices*. Elsevier. (2004).
- [21] A. Metcalfe and N. O’Connell and J. Warren. Interlaced processes on the circle. *Ann. Inst. H. Poincaré Probab. Statist.* 45, 4 (2009) 1165–1184.
- [22] A. Nica and R. Speicher Lectures on the Combinatorics of Free Probability. *LMS Lecture Note Series* (2006).
- [23] Ph.D. thesis of E. Nordenstam. Interlaced particles in tilings and random matrices. *KTH*. (2009).
- [24] A. Okounkov and N. Reshetikhin. The birth of a random matrix. *Mosc. Math. J.* 6,3 (2006) 553–566.
- [25] L. A. Pastur and M. Shcherbina. Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles. *J. Stat. Phys.* 86 (1997) 109–147.
- [26] W. Rudin. *Real and Complex Analysis*, Third edition, McGraw-Hill, 1987.
- [27] A. Soshnikov. Determinantal random point fields. *Russian Mathematical Surveys*. 55 (2000) 923–975.
- [28] G. Szegő. *Orthogonal Polynomials*. American Mathematical Society. (1939).
- [29] D. Voiculescu. A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Internat. Math. Res. Notices*. 1 (1998), 41–63.
- [30] F. Xu. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 190,2 (1997), 287–307.
- [31] J. Warren. Dyson’s Brownian motions, intertwining and interlacing. *Electron. J. Probab.* 12,19 (2007), 573–590.

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